

Online Linear Discrepancy

Mitchel T. Keller, Noah Streib, and William T. Trotter

School of Mathematics
Georgia Institute of Technology

SIAM DM10
16 June 2010



- 1 Linear Discrepancy
- 2 Online Algorithms
- 3 Online Linear Discrepancy
 - Lower Bounds
 - Upper Bounds
 - Representations
- 4 Open Problems

Linear Discrepancy Defined

Definition

The *linear discrepancy* of a linear extension L of a poset \mathbf{P} is

$$\text{ld}(\mathbf{P}, L) = \max_{x \parallel_P y} |h_L(y) - h_L(x)|.$$

Definition (Tanenbaum, Trenk, and Fishburn 2001)

The *linear discrepancy* of a poset \mathbf{P} is $\text{ld}(\mathbf{P}) = \min_{L \in \mathcal{E}(\mathbf{P})} \text{ld}(\mathbf{P}, L)$ if \mathbf{P} is not a chain. If \mathbf{P} is a chain, $\text{ld}(\mathbf{P}) = 0$.

Facts About Linear Discrepancy

Theorem (Fishburn, Tanenbaum, and Trenk 2001)

If \mathbf{P} is a poset with incomparability graph \mathbf{G} , then

$$\text{ld}(\mathbf{P}) = \text{bw}(\mathbf{G}).$$

Facts About Linear Discrepancy

Theorem (Fishburn, Tanenbaum, and Trenk 2001)

If \mathbf{P} is a poset with incomparability graph \mathbf{G} , then

$$\text{ld}(\mathbf{P}) = \text{bw}(\mathbf{G}).$$

Theorem (Kloks, Kratsch, Müller 1999)

$$\text{ld}(\mathbf{P}, L) \leq 3 \text{ld}(\mathbf{P})$$

Facts About Linear Discrepancy

Theorem (Fishburn, Tanenbaum, and Trenk 2001)

If \mathbf{P} is a poset with incomparability graph \mathbf{G} , then

$$\text{ld}(\mathbf{P}) = \text{bw}(\mathbf{G}).$$

Theorem (Kloks, Kratsch, Müller 1999)

$$\text{ld}(\mathbf{P}, L) \leq 3 \text{ld}(\mathbf{P})$$

Lemma (TTF 2001, Rautenbach 2005)

- (1) *Linear discrepancy is monotonic.*
- (2) $\text{ld}(\mathbf{P}) \geq \text{width}(\mathbf{P}) - 1$
- (3) $\Delta(\mathbf{P})/2 \leq \text{ld}(\mathbf{P}) \leq 2\Delta(\mathbf{P}) - 2$

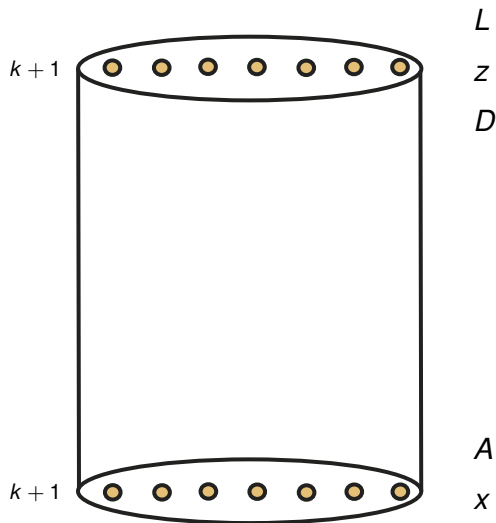
- (1) *Builder* presents poset one point at a time
- (2) *Assigner* makes irrevocable decision about where to insert it into a linear extension.

- (1) *Builder* presents poset one point at a time
- (2) *Assigner* makes irrevocable decision about where to insert it into a linear extension.
- (3) **Variant:** Builder is required to provide a representation (interval, unit interval, etc.)

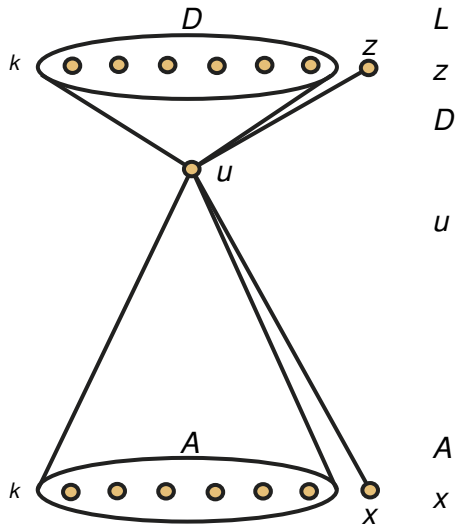
Lemma (K, Streib, Trotter)

For each $k \geq 1$, Builder can construct an **interval order** \mathbf{P} with $\text{ld}(\mathbf{P}) = k$ so that Assigner must form a linear extension L with $\text{ld}(\mathbf{P}, L) = 3k - 1$.

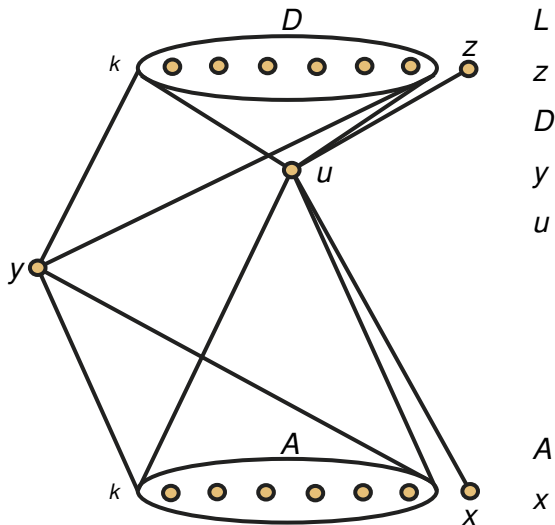
Lower Bound for General Posets



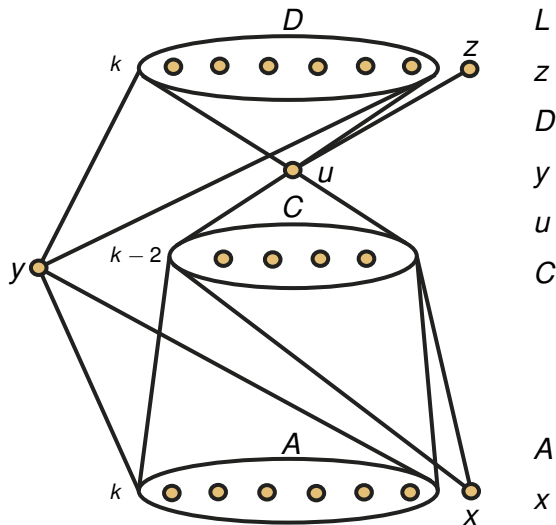
Lower Bound for General Posets



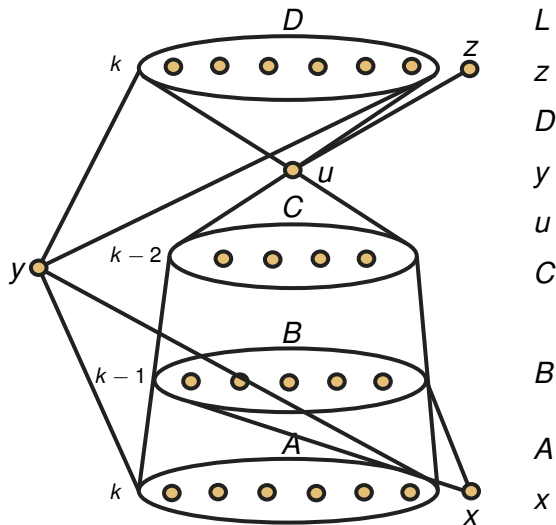
Lower Bound for General Posets



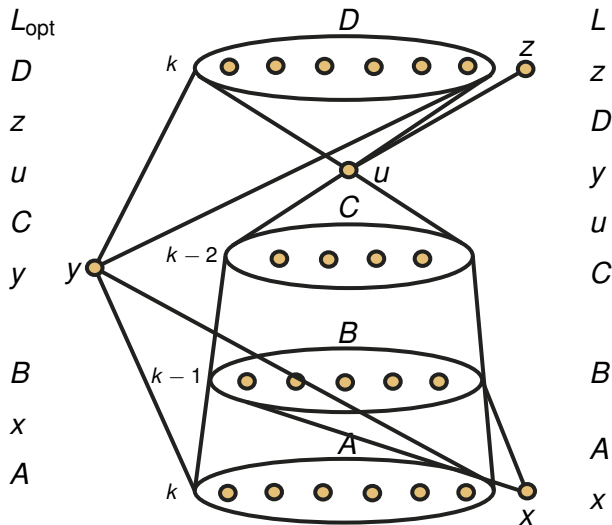
Lower Bound for General Posets



Lower Bound for General Posets



Lower Bound for General Posets



Lemma (KST)

For each $k \geq 1$, Builder can construct a semiorder \mathbf{P} with $\text{ld}(\mathbf{P}) = k$ so that Assigner must form a linear extension L with $\text{ld}(\mathbf{P}, L) = 2k$.

Theorem (KST)

If Builder constructs a poset \mathbf{P} with $\text{ld}(\mathbf{P}) = k \geq 1$ and Assigner uses algorithm \mathcal{A} to assemble a linear extension L of \mathbf{P} , then

- (1) $\text{ld}(\mathbf{P}, L) \leq 2k$ if \mathbf{P} is a semiorder and*
- (2) $\text{ld}(\mathbf{P}, L) \leq 3k - 1$ otherwise.*

First Attempts at Algorithms

- \mathcal{M} : Insert point in middle of its allowable range in existing linear extension.
- \mathcal{G} : Insert point to minimize $\text{ld}(\mathbf{P}, L)$ for updated linear extension.

Definition

A pair (x, y) of incomparable points is a *critical pair* if

(1) $D(x) \subseteq D(y)$

(2) $U(y) \subseteq U(x)$

Definition

A pair (x, y) of incomparable points is a *critical pair* if

- (1) $D(x) \subseteq D(y)$
- (2) $U(y) \subseteq U(x)$

Lemma (K, Young 2010)

- (1) If x and y are incomparable and $h_L(y) - h_L(x) = \text{ld}(\mathbf{P}, L)$, then (x, y) is a critical pair.
- (2) There exists an optimal linear extension L of \mathbf{P} so that if (x, y) is a critical pair reversed by L , then (y, x) is also a critical pair.

An Optimal Online Linear Discrepancy Algorithm

In four words:



An Optimal Online Linear Discrepancy Algorithm

In four words:

Reverse few critical pairs.



An Optimal Online Linear Discrepancy Algorithm

In four words:

Reverse few critical pairs.

A bit more detail

Builder presents new point x . Assigner uses algorithm \mathcal{A} :

- Identify one-way critical pairs (x, u) and (v, x)
- u_0 the L -smallest u , v_0 the L -largest v
- None of either type: x in any legal position
- Only (x, u) : x below u_0
- Only (v, x) : x above v_0
- $v_0 < u_0$ in L : x any legal position between v_0 and u_0

An Optimal Online Linear Discrepancy Algorithm

In four words:

Reverse few critical pairs.

A bit more detail

Builder presents new point x . Assigner uses algorithm \mathcal{A} :

- Identify one-way critical pairs (x, u) and (v, x)
- u_0 the L -smallest u , v_0 the L -largest v
- None of either type: x in any legal position
- Only (x, u) : x below u_0
- Only (v, x) : x above v_0
- $v_0 < u_0$ in L : x any legal position between v_0 and u_0
- $u_0 < v_0$ in L ?

An Optimal Online Linear Discrepancy Algorithm

In four words:

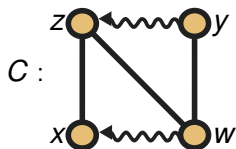
Reverse few critical pairs.

A bit more detail

Builder presents new point x . Assigner uses algorithm \mathcal{A} :

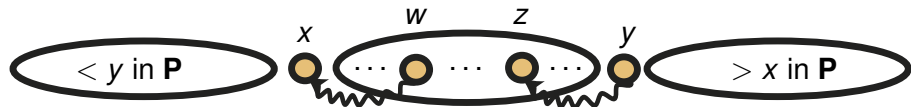
- Identify one-way critical pairs (x, u) and (v, x)
- u_0 the L -smallest u , v_0 the L -largest v
- None of either type: x in any legal position
- Only (x, u) : x below u_0
- Only (v, x) : x above v_0
- $v_0 < u_0$ in L : x any legal position between v_0 and u_0
- $u_0 < v_0$ in L ? x anywhere between u_0 and v_0

A Look At Why It Works



Lemma (KST)

A never assembles a linear extension L with a copy of C such that $x < w < z < y$ in L , all points less than x in L are less than y in \mathbf{P} , and all points greater than y in L are greater than x in \mathbf{P} .



Theorem (KST)

If Builder constructs a poset \mathbf{P} with $\text{ld}(\mathbf{P}) = k \geq 1$ and Assigner uses algorithm \mathcal{A} to assemble a linear extension L of \mathbf{P} , then

- (1) $\text{ld}(\mathbf{P}, L) \leq 2k$ if \mathbf{P} is a semiorder and*
- (2) $\text{ld}(\mathbf{P}, L) \leq 3k - 1$ otherwise.*

Theorem (KST)

If Builder is required to present an interval representation of an interval order \mathbf{P} with $\text{ld}(\mathbf{P}) = k \geq 1$ and Assigner constructs a linear extension L ordering by left endpoints, then

- (1) $\text{ld}(\mathbf{P}, L) = k$ if \mathbf{P} is a semiorder and Builder presents unit intervals and*
- (2) $\text{ld}(\mathbf{P}, L) \leq 2k$ otherwise.*

- Online linear discrepancy
 - No representation, interval orders without $1 + 4$?
 - Proper representation for semiorders instead of unit?
 - Up-growing?

- Online linear discrepancy
 - No representation, interval orders without $\mathbf{1} + \mathbf{4}$?
 - Proper representation for semiorders instead of unit?
 - Up-growing?
- Is $\text{ld}(\mathbf{P}) \leq \left\lfloor \frac{3\Delta(\mathbf{P}) - 1}{2} \right\rfloor$?
 - Known for width 2, disconnected, interval orders.
 - Best general upper bound is $2\Delta(\mathbf{P}) - 2$.

- Online linear discrepancy
 - No representation, interval orders without $\mathbf{1} + \mathbf{4}$?
 - Proper representation for semiorders instead of unit?
 - Up-growing?
- Is $\text{ld}(\mathbf{P}) \leq \left\lfloor \frac{3\Delta(\mathbf{P}) - 1}{2} \right\rfloor$?
 - Known for width 2, disconnected, interval orders.
 - Best general upper bound is $2\Delta(\mathbf{P}) - 2$.
- Is $\text{ld}(\mathbf{P}) \geq \dim(\mathbf{P})$ for $\dim(\mathbf{P}) \geq 5$?