RESEARCH STATEMENT

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Overview

Broadly, my research interests are in combinatorial mathematics. My research has focused on the combinatorics of partially ordered sets and online algorithms, but I am also interested in extremal problems in other areas of combinatorics. As a graduate student, I was fortunate to develop collaborations with other early career mathematicians, and I am working to further develop those relationships while expanding my network of collaborators through my postdoctoral position. This overview gives a summary of my research interests for those who are not specialists in combinatorics. The subsequent sections provide more detailed information about my research, including problems on which I intend to work in the near future.

Combinatorics of Partially Ordered Sets. Partially ordered sets (or posets) provide combinatorial models for comparison relationships. One example of a poset is the lattice of subsets of a finite set, where the partial order is given by subset inclusion. Many questions about posets, including the two discussed in this statement, involve properties of their linear extensions. A *linear extension* of a partial order P is a total order L on the same ground set that respects P, i.e., if $x \leq y$ in P, then $x \leq y$ in L.

One poset parameter on which I have worked is *linear discrepancy*. The problem of linear discrepancy arises naturally when considering linear extensions of a partial order. If incomparable elements are placed far apart, users in applied settings may create an implicit comparison between them. This comparison unfairly biases against the element that is placed lower, suggesting that it is smaller in the original partial order. Linear discrepancy attempts to quantify this unfairness by measuring the largest distance between incomparable elements in a linear extension and minimizing that distance over all linear extensions.

Part of my linear discrepancy research involved completing the characterization of the posets with linear discrepancy 2. We provided a complete list of the forbidden subposets for such posets. My research has also provided upper bounds on a poset's linear discrepancy in terms of the maximum number of elements of the ground set to which any element is incomparable. This work partially addresses one of the original questions asked about linear discrepancy by providing tight bounds for two classes of posets—disconnected posets and interval orders.

The *linear extension diameter* of a poset measures how different two linear extensions can be. Specifically, the distance between linear extensions L_1 and L_2 is the number of pairs of incomparable elements x, y for which $x < y$ in L_1 and $y < x$ in L_2. We say such an incomparable pair is *reversed* by L_1 and L_2. The linear extension diameter of a poset is the maximum distance between two of its linear extensions. Some posets have linear extension diameter equal to the number of incomparable pairs, but in general this is not the case. In fact, we have recently shown that it is not always possible to find a pair of linear extensions that reverse a constant fraction of the total number of incomparable pairs. We have also begun investigating the relationship between linear extension diameter and long-studied poset parameters such as width and dimension.

Online algorithms. Another of my research interests is online algorithms. The traditional framework for algorithms applied to discrete structures provides the algorithm with complete information about the structure throughout execution. In contrast, an online algorithm receives information about a structure one piece at a time and makes an irrevocable decision at each stage. For some applications, the online algorithm framework more accurately represents how a problem must be solved, since it is impossible to supply complete information at the outset. The performance of an online algorithm is generally compared to that of an optimal, perhaps inefficient, algorithm in the traditional framework.

My research has included investigation of online algorithms for linear discrepancy. We have obtained a best possible online algorithm for linear discrepancy on arbitrary posets and shown that when restricted to a particular subclass of posets (semiorders) it remains best possible, although with a stronger performance guarantee. There have recently been several new results related to another online problem for posets, and it appears that these new ideas may lead to resolving a problem that has been open for more than 30 years.

Stanley depth: A connection to commutative algebra. Nearly 30 years ago, R.P. Stanley made a conjecture involving two parameters of modules over a polynomial ring over a field. Little progress had been made on Stanley's conjecture until recently, when a team of researchers identified a way to approach some cases of the problem through a partitioning problem on certain posets. This discovery has led to interesting combinatorics in addition to progress on Stanley's conjecture. There are a number of questions yet to be resolved in this area through deeper collaboration between combinatorists and algebraists. To date, the algebraists and combinatorists working in this area have tended to work separately. In the coming years, I plan to continue collaborating with combinatorists interested in these problems and to develop new collaborations with the algebraists who are best positioned to identify the most important questions to answer.

1. COMBINATORICS OF PARTIALLY ORDERED SETS

Linear discrepancy. One poset parameter in which I am interested is *linear discrepancy*. When forming a linear extension L of a poset $\mathbf{P}$, there is no requirement on the ordering of incomparable elements x and y in L. In fact, x and y might be placed very far apart in L, which can unfairly suggest that x and y are comparable in $\mathbf{P}$. In [25], Tanenbaum, Trenk, and Fishburn introduced linear discrepancy as a way to quantify the minimum unfairness a linear extension of a given poset can exhibit. To define linear discrepancy precisely, let $h_L(x)$ be the position of x in L (with the L-least element having position 1). If all elements of $\mathbf{P}$ are pairwise comparable, the linear discrepancy of $\mathbf{P}$ is then defined to be 0, and $\mathrm{ld}(\mathbf{P}) = \min_L \max_{x\|_{\mathbf{P}} y} |h_L(x) - h_L(y)|$ otherwise. Here the minimum is taken over all linear extensions of $\mathbf{P}$, and the maximum is taken over all incomparable pairs. In [8], the same authors showed that $\mathrm{ld}(\mathbf{P})$ is equal to the bandwidth of $\mathbf{P}$'s incomparability graph, thereby showing that the decision problem $\mathrm{ld}(\mathbf{P}) \leq k$ is **NP**-complete for general posets by a result of Kloks, Kratsch, and Müller in [19].

In [25], Tanenbaum et al. provided an elegant forbidden subposet characterization of the posets of linear discrepancy 1. They then asked if the posets of linear discrepancy 2 could be similarly characterized. In [22], Rautenbach conjectured that a small list of forbidden subposets characterized the posets of linear discrepancy 2. Extending the work of Howard, Chae, Cheong, and Kim in [12], D.M. Howard, S.J. Young, and I provided a forbidden subposet characterization of the posets of linear discrepancy 2 in [13]. The forbidden subposets include an infinite family and a small number of exceptional posets. The members of the infinite family on n points are all derived from a single n-point poset by the removal of certain comparability relations.

For most monotonic combinatorial parameters, removing a single point can cause only a small change in the parameter's value. However, for linear discrepancy it is possible to remove a point and create a poset with much smaller linear discrepancy. For example, the poset consisting of a single point incomparable to a chain of n points, i.e., n points forming a linear order, has linear discrepancy $\lceil n/2 \rceil$. However, removing the isolated point results in a poset of linear discrepancy 0. Despite this difficulty, S.J. Young and I were able to prove a weaker, but still useful, point removal property in [17]. We showed that for every poset $\mathbf{P}$, there exists a point which can be deleted without decreasing the linear discrepancy by more than one. We also proved two useful results about the relationship between linear discrepancy and critical pairs, which feature prominently in the study of order dimension.

We employed these results to address a question from the original paper on linear discrepancy. Let $\Delta(\mathbf{P})$ be the maximum number of elements of $\mathbf{P}$ to which any element is incomparable. In [25], Tanenbaum et al. asked if every poset satisfied $\mathrm{ld}(\mathbf{P}) \leq \lfloor (3\Delta(\mathbf{P}) - 1)/2 \rfloor$. S.J. Young and I answered this question in the affirmative for disconnected posets and interval orders in [17]. Our result for interval orders is similar to Brooks' Theorem in that $\mathrm{ld}(\mathbf{P}) \leq \Delta(\mathbf{P})$ for an interval order $\mathbf{P}$ with equality if and only if $\mathrm{width}(\mathbf{P}) = \Delta(\mathbf{P}) + 1$. (The width of a poset is the size of a maximum set of pairwise incomparable elements.) We also showed that this result is best possible by constructing a family of width 2 interval orders for which $\mathrm{ld}(\mathbf{P}) = \Delta(\mathbf{P}) - 1$. Milans [20] subsequently used probabilistic techniques to show the existence of posets of height two with linear discrepancy greater than the proposed bound (and close to the best known general upper bound of $2\Delta(\mathbf{P}) - 2$). In light of Milans' result, it becomes interesting to determine best possible bounds on $\mathrm{ld}(\mathbf{P})$ in terms of $\Delta(\mathbf{P})$ for other classes of posets.

Another question of interest involves the relationship between linear discrepancy and dimension. The dimension of a poset $\mathbf{P} = (X, \leq_P)$, denoted $\dim(\mathbf{P})$, is the smallest t such that there exist linear extensions $L_1, \ldots, L_t$ such that $x \leq_P y$ if and only if $x \leq_{L_i} y$ for $1 \leq i \leq t$. Dimension is a much studied poset parameter, and it is not difficult to show that $\mathrm{ld}(\mathbf{P}) \geq \dim(\mathbf{P}) - 1$. In [25], Tanenbaum et al. state the following conjecture of Brightwell and Trotter: If $n = \dim(\mathbf{P}) \geq 5$, then $\mathrm{ld}(\mathbf{P}) \geq \dim(\mathbf{P})$, and if equality holds, then $\mathbf{P}$ must contain a copy of a specific n-dimensional poset on $2n$ points known as the standard example $\mathbf{S}_n$. In [26], Trotter gives an example of a poset with $\dim(\mathbf{P}) = 4$ but $\mathrm{ld}(\mathbf{P}) = 3$. However, the construction does not generalize, and there has been no progress toward resolving the conjecture. The literature on linear discrepancy is now developed to the point that serious work on this conjecture is possible.

Linear extension diameter. In [7], Felsner and Reuter introduced the *linear extension diameter*, denoted $\mathrm{led}(\mathbf{P})$, of a poset. It is the maximum over all pairs of linear extensions L_1, L_2 of the number of incomparable pairs x, y with $x <_{L_1} y$ and $y <_{L_2} x$. Brightwell and Massow showed in [5] that if $\mathrm{width}(\mathbf{P}) \leq 3$, then $\mathrm{led}(\mathbf{P})$ can be determined in polynomial time. However, they also showed that determining $\mathrm{led}(\mathbf{P})$ is **NP**-complete in general. It is natural to ask how close the linear extension diameter of $\mathbf{P}$ can be to $\mathrm{inc}(\mathbf{P})$, the total number of (unordered) incomparable pairs of $\mathbf{P}$. G.R. Brightwell and I approached this in [4] by using probabilistic techniques to find a family of posets $\mathbf{P}_k$ of width k for which there exist constants c, c' such that with high probability $\mathrm{inc}(\mathbf{P}_k) \geq ck^2 \log^2 k$ and $\mathrm{led}(\mathbf{P}) \leq c'k^2 \log k$. In studying this question, we introduced the reversal ratio of $\mathbf{P}$ as $RR(\mathbf{P}) = \mathrm{led}(\mathbf{P}) / \mathrm{inc}(\mathbf{P})$. Put in this notation, our family of examples has reversal ratio at most $33/\log k$ with high probability. However, it remains open to determine if there is a function $f(n)$ such that $RR(\mathbf{P}) \geq f(n)$ for all posets $\mathbf{P}$ with $g(\mathbf{P}) = n$ where $g(\mathbf{P})$ is some "reasonable" function of $\mathbf{P}$, such as the number of points or width.

We also considered ways to bound $RR(\mathbf{P})$ in terms of long-studied poset properties such as dimension and width. There is a constant c'' so that the examples $\mathbf{P}_k$ mentioned previously have dimension at least $c''k$ with high probability. Thus, $RR(\mathbf{P}_k) \leq C/\log d$ with high probability. We also showed that $RR(\mathbf{P}) \geq 2/d$ if $\dim(\mathbf{P}) = d$. However for $d > 3$, we do not know the smallest that $RR(\mathbf{P})$ can be for this class of posets. We suspect that it is closer to $C/\log d$. To discuss bounding the reversal ratio in terms of width, let $h_{RR}(w) = \inf\{RR(\mathbf{P})\colon \text{width}(\mathbf{P} \leq w\}$. Arguments involving dimension show that $h_{RR}(3) \geq 2/3$, and we have constructed a family of examples showing that $h_{RR}(3) \leq 5/6$. However, Brightwell and I conjecture that $h_{RR}(3) = 3/4$ and that a family of examples given in [4] is (essentially) extremal. Addressing questions such as these should give further insight into the nature of linear extension diameter.

2. Online Algorithms

It is convenient to think of an online combinatorial problem as a game between two players called Builder and Assigner. For online problems involving posets, Builder presents the points of the poset one at a time and at each step informs Assigner of the relations between the previously-presented points and the new point. This information must be consistent with that previously presented, e.g., transitivity cannot make two previously-incomparable points comparable when a new point is presented. Assigner then makes an irrevocable decision such as where to place the new point in a linear extension or into which chain of a chain partition the new point should be placed.

Typically, the analysis of an online algorithm involves comparing the algorithm's output with that of an optimal offline algorithm, i.e., one operating in the traditional framework. For online linear discrepancy, Assigner maintains a linear extension L of the poset Builder has presented, striving to keep the linear discrepancy of L as small as possible. Kloks, Kratsch, and Müller showed in [19] that no linear extension has linear discrepancy larger than $3\,\text{ld}(\mathbf{P})$, so any algorithm Assigner uses will differ from optimal by a factor of at most 3. This contrasts with the online graph bandwidth problem, which Board considered in [2]. He showed that for arbitrary graphs, Builder can force Assigner to construct a permutation of the vertex set with bandwidth far from the optimal value.

N. Streib, W.T. Trotter, and I considered the problem of online algorithms for linear discrepancy in [15]. We gave an online algorithm guaranteed to place each incomparable pair at distance at most $3k-1$ for any poset of linear discrepancy k. We also showed that this is best possible for every positive integer k, even when restricted to *interval orders*, i.e., posets for which there exists a way to assign each element x to an interval $I(x) \subset \mathbb{R}$ such that $x < y$ if and only if $I(x)$ lies completely left of $I(y)$. A *semiorder* is an interval order in which all intervals have unit length. When restricted to semiorders of linear discrepancy k, the algorithm maintains $|h_L(x) - h_L(y)| \leq 2k$, and this is best possible. While our work provided optimal results for general online linear discrepancy algorithms, this problem has not been investigated for the up-growing orders framework introduced by Felsner in [6]. For up-growing posets, each new point must be maximal, i.e., either incomparable to or greater than each previously-presented point. Since the known examples for online linear discrepancy rely on presenting a large number of non-maximal points, it is possible a better result can be obtained in this framework.

A longstanding open question in online algorithms is the problem of partitioning a poset into chains online. (This is related to, but not equivalent to, coloring an incomparability graph online.) Dilworth's Theorem guarantees that a poset of width w can be partitioned offline into w but no fewer chains. An argument attributed to Szemerédi shows that any online algorithm can be forced to use at least $\binom{w+1}{2}$ chains to partition a poset of width w. For over 25 years, the best known algorithm was Kierstead's in [18], which could only be shown to require no more than $(5^w - 1)/4$ chains on a poset of width w. In [3], Bosek and Krawczyk recently gave an online algorithm requiring at most $w^{16 \lg w}$ chains to partition a poset of width w. Their work has spurred increased interest in this problem, with progress being made on questions involving restricted classes of posets. While most believe Szemerédi's lower bound to not be optimal, it is also widely thought that an online algorithm only requiring a number of chains polynomial in w exists. New developments in this area may make it possible to finally resolve this problem.

3. Interval Partitions and Stanley Depth

In this section, let K be a field and $S = K[x_1, \ldots, x_n]$. R.P. Stanley introduced the notion of the *Stanley depth* of an S-module M, denoted $\operatorname{sdepth}(M)$, in purely algebraic terms in [24]. He conjectured that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$. Progress had been made on Stanley's conjecture in special cases, but a major breakthrough came recently when Herzog, Vladoiu, and Zheng showed in [11] that for monomial ideals $J \subset I \subset S$, the Stanley depth of I/J can be determined by partitioning a poset defined by the generators of I and J into intervals $[x, y] = \{z \in \mathbf{P} \colon x \leq z \leq y\}$. The posets in question are all finite subposets of $\mathbb{N}^n$, with a monomial corresponding to the n-tuple of its powers of the variables. If I is squarefree, the poset associated to I can be viewed as a subposet of the Boolean algebra of subsets of $[n] := \{1, 2, \ldots, n\}$. These algebraic questions therefore give rise to interesting combinatorial partitioning problems.

Herzog et al. asked if $\operatorname{sdepth}(x_1, \ldots, x_n) = \lceil n/2 \rceil$ in [11]. From the combinatorial perspective, this question has an appealing restatement: Can the poset consisting of the nonempty subsets of $[n]$ be partitioned into a collection $\mathcal{P}$ of intervals so that for all $[X, Y] \in \mathcal{P}$, $|Y| \geq \lceil n/2 \rceil$? In [1], Cs. Biró, D.M. Howard, W.T. Trotter, S.J. Young, and I showed that the answer to this question is "yes," and gave two proofs yielding nonisomorphic partitions. The result we proved is stronger than needed for the algebraic result and provides a structural view of the poset of nonempty subsets of $[2k+1]$. (The result for $n = 2k+2$ follows by a parity argument.)

Shen showed in [23] that the algebraic version of our result can be extended to show that the Stanley depth of a complete intersection monomial ideal minimally-generated by m monomials is $n - \lfloor m/2 \rfloor$. Shen also proved lower bounds for the Stanley depth of 3- and 4-generated squarefree monomial ideals that are not complete intersection monomial ideals. He then asked if every m-generated squarefree monomial ideal in S has Stanley depth at least $n - \lfloor m/2 \rfloor$. S.J. Young and I answered this question in the affirmative in [16], where we offered a short proof that encompassed Shen's results for 3- and 4-generated squarefree monomial ideals. We did this by focusing on the combinatorial view, while Shen had passed repeatedly between the language of posets and that of ideals.

To generalize the result of [1], it is natural to consider the problem of finding an interval partition of the poset consisting of all subsets $T \subseteq [n]$ with $|T| \geq d > 1$. This poset corresponds to the *squarefree Veronese ideal of degree d* in $K[x_1, \ldots, x_n]$. For $d > 1$, we also want intervals $[X, Y]$ for which $|Y|$ is bounded below, but obviously $\lceil n/2 \rceil$ is not attainable. For $d \leq \lfloor n/2 \rfloor$, a counting argument suggests a conjecture that $\operatorname{sdepth} I_{n,d} = \left\lfloor \binom{n}{d+1} / \binom{n}{d} \right\rfloor + d$. In [14], Y.-H. Shen, N. Streib, S.J. Young, and I showed that the conjectured formula for $\operatorname{sdepth} I_{n,d}$ holds for $n = cd + (c-1)$ when $c \leq 4$. This gives a formula for $\operatorname{sdepth} I_{n,d}$ that holds for n and d with $1 \leq d \leq n < 5d + 4$. Our proof uses a construction that generalizes the constructive proof in [1] and involves lemmas about this construction that suggest the conjecture to be true in general. In [9], Ge, Lin, and Shen have advanced the state of the art to confirm the conjecture for $d \in \Omega(n^{2/3})$.

The *Veronese ideal of degree d* in S is the monomial ideal generated by *all* monomials of degree d. In [10], Ge, Lin, and Wang established a relationship between the Hilbert depth of Veronese ideals and squarefree Veronese ideals. They also conjectured that the same relationship holds for the Stanley depth of these classes of ideals. No progress has been made on this interesting conjecture, however. Insight into these types of problems has the potential to develop techniques that can be used to attack other open problems about the subset lattice.

The key to resolving Stanley's conjecture for all monomial ideals could rest in a question posed by Rauf in [21]. Specifically, Rauf asked if $\operatorname{sdepth} I > \operatorname{sdepth} S/I$ for all monomial ideals $I \subset S$. This question has a nice combinatorial interpretation, as the poset corresponding to I and the poset corresponding to S/I are disjoint and their union is $\{c \in \mathbb{N}^n : c \leq g\}$ for an appropriately-chosen g. Although providing an affirmative answer to this question would not entirely resolve Stanley's conjecture for monomial ideals, it would reduce it to the case of cyclic S-modules.

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