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Reversal Ratio

Reversal Ratio and Linear Extension Diameter

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Outline

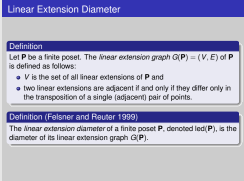
Definition

Let **P** be a finite poset. The *linear extension graph* $G(\mathbf{P}) = (V, E)$ of **P** is defined as follows:

- V is the set of all linear extensions of **P** and
- two linear extensions are adjacent if and only if they differ only in the transposition of a single (adjacent) pair of points.

Another way to think about led is as the largest number of *unordered* incomparable pairs that can appear in opposite order in a pair of linear extensions.

NP-complete in general. Polynomial via dynamic programming for width 3.



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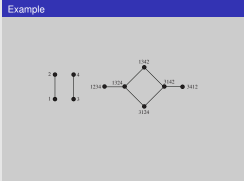
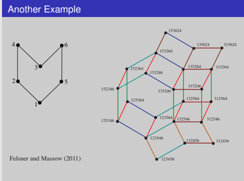


Here we've got four incomparable pairs and the diameter is four.



Edges are colored in such a way that all edges colored a given color correspond to swapping the same incomparable pair. (Red = 4–6, Green = 2–3) Now we've got seven pairs but the led is six.

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Reversal Ratio
Linear Extension Diameter
Reversal Ratio

Definition

Let **P** be a poset and L_1, L_2 linear extensions of **P**. We define the *reversal ratio of the pair* (L_1, L_2) as

$$RR(\mathbf{P}; L_1, L_2) = \frac{\text{dist}(L_1, L_2)}{\text{inc}(\mathbf{P})}.$$

The *reversal ratio of* **P** is

$$RR(\mathbf{P}) = \frac{\text{led}(\mathbf{P})}{\text{inc}(\mathbf{P})} = \max_{L_1, L_2} RR(\mathbf{P}; L_1, L_2).$$

Bounding Reversal Ratio

- $RR(\mathbf{P}) = 1$ if and only if $\dim(\mathbf{P}) = 2$.

- Every term is at most $\text{led}(\mathbf{P})$. Each incomparable pair contributes to the sum at least $d - 1$ times. (Minimum is if it's in one order in $d - 1$ linear extensions and the other only once.)
- Take the pair of linear extensions that reverses the large antichain.

Bounding Reversal Ratio

- $RR(\mathbf{P}) = 1$ if and only if $\dim(\mathbf{P}) = 2$.
- If $\dim(\mathbf{P}) = d$ and $\mathcal{R}$ is a realizer of $\mathbf{P}$, then

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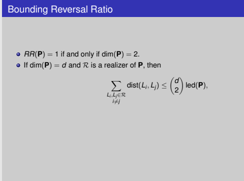

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  so $RR(\mathbf{P}) \geq 2/d$.
- $\mathrm{width}(\mathbf{P}) = c|\mathbf{P}|$ for $c > 0$ implies $RR(\mathbf{P})$ is large.

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- Is there a $c > 0$ such that $RR(\mathbf{P}) \geq c$ for all $\mathbf{P}$?

- Large dimension, many incomparable pairs, width not too big leads to thinking about $\mathbf{2}^n$.
- $\text{inc}(2^n) \sim 2^{2n-1}$ and Felsner-Massow's work implies that $(\mathbf{P}) \geq 1/2$ for downset lattice of 2-dimensional posets.
- Note that people seem to keep forgetting that the answer to this questio is "No."
- Brightwell's original example uses random graph orders. ($G_{n,p}$ for suitable p and let $i < j$ if they are joined by an edge. Take transitive closure.)

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  - Unpublished example difficult to analyze.

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How small can $RR(\mathbf{P})$ be?

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Theorem (BK 2011+)

For every sufficiently large positive integer k, there exists a poset $\mathbf{P}_k$ of width k with $RR(\mathbf{P}_k) \leq C/\log k$.

Doubling Property

Definition

Let $\mathbf{G} = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = k$. We say that $\mathbf{G}$ has the doubling property if for every $Y \subset A$ with $|Y| \leq k/3$, $|N(A)| \geq 2|A|$.

1. Doubling property is an example of vertex expansion property, so such $\mathbf{G}$ is a $(1/3, 2)$ vertex-expander. Explicit constructions of $(1/3, 2)$ vertex-expanders exist: Ramanujan graphs, but construction is very complex.
2. Probability at least $1 - 2/k^5$.

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Lemma
Let $G_d(A, B)$ be a random d-regular bipartite graph on vertex sets A and B of size k, chosen according to the configuration model. For each $d \geq 10$ and k sufficiently large, $G_d(A, B)$ has the doubling property with high probability.

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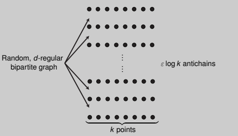
Construction of $\mathbf{P}_k$



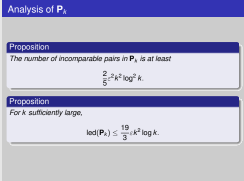
Analysis of $\mathbf{P}_k$

Proposition

The number of incomparable pairs in $\mathbf{P}_k$ is at least

$$\frac{2}{5}\varepsilon^2 k^2 \log^2 k.$$

- Comes straight from counting how many elements a fixed element can be comparable to (geometric series).
- This is where the doubling property comes into play. When taking sets size t at different levels and trying to push them past each other, you'll be blocked by a comparability that the doubling property ensures.



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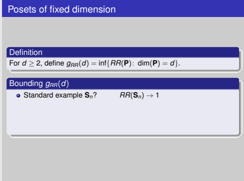
For $d \geq 2$, define $g_{RR}(d) = \inf\{RR(\mathbf{P})\colon \dim(\mathbf{P}) = d\}$.

Bounding $g_{RR}(d)$

- Standard example $\mathbf{S}_n$?

First observation is that $g_{RR}(2) = 1$. Recall from earlier also that $g_{RR}(d) \geq 2/d$.

Felsner-Massow gives an exact formula for $\operatorname{led}(\mathbf{n}^d)$



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- $g_{RR}(3) = 2/3$ by considering $\mathbf{n}^3$
- $1/2 \leq g_{RR}(4) \leq 4/7$

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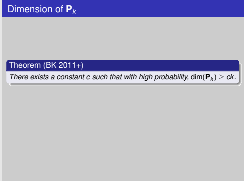
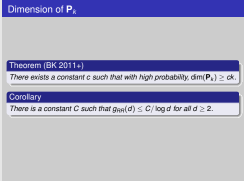


The proof counts the number of incomparable max-min pairs and then uses the doubling property to estimate how many a pair of linear extensions could reverse. This allows us to give a lower bound on the size of a realizer of $\mathbf{P}_k$.



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Width 2 is not interesting, since dim $\leq$ width. Inequality follows from this and is only valid for 3 because of this.

The 5/6 bound comes from stacking standard examples. Add six new critical pairs (three horizontal, three vertical) and can reverse five of them.

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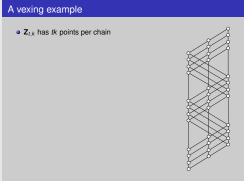
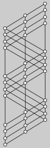


We've got the reversal ratio is at least $3/4$ for this family of posets, but could it be higher? $k = 3$ can be through a bit of cheating.

A vexing example

- $\mathbf{Z}_{t,k}$ has tk points per chain
- $\mathrm{inc}(\mathbf{Z}_{t,k}) = (6t-3)\binom{k}{2}$



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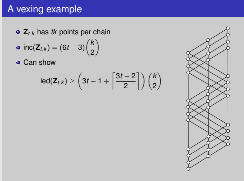

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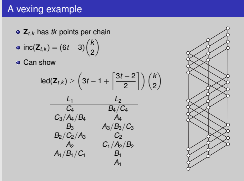

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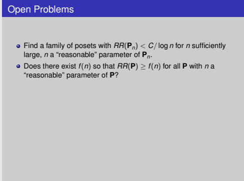
- Reasonable: number of points, width, etc.
- Dimension question, we think it's more likely to be closer to the upper bound than the lower.



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Conjecture (BK)

$$h_{RR}(3) = 3/4$$

- Reasonable: number of points, width, etc.

Thank You

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Web: `http://rellek.net`