Reversal Ratio and Linear Extension Diameter

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Outline

Linear Extension Diameter

Definition

Let **P** be a finite poset. The *linear extension graph* $G(\mathbf{P}) = (V, E)$ of **P** is defined as follows:

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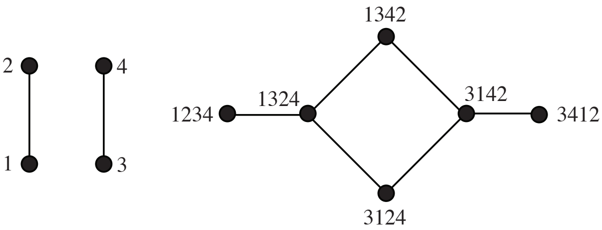
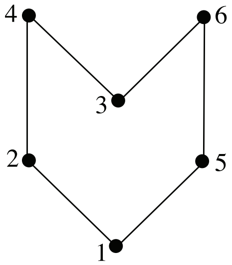
Definition (Felsner and Reuter 1999)

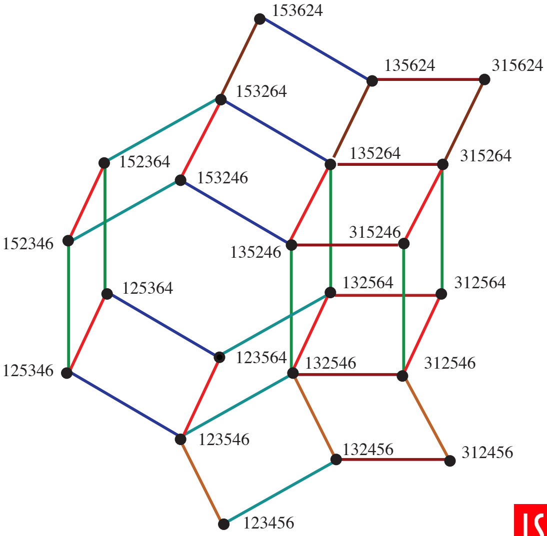
The *linear extension diameter* of a finite poset $\mathbf{P}$, denoted $\text{led}(\mathbf{P})$, is the diameter of its linear extension graph $G(\mathbf{P})$.

Example



Another Example





Felsner and Massow (2011)

Reversal Ratio

Definition

Let $\mathbf{P}$ be a poset and L_1, L_2 linear extensions of $\mathbf{P}$. We define the *reversal ratio of the pair* (L_1, L_2) as

$$RR(\mathbf{P}; L_1, L_2) = \frac{\text{dist}(L_1, L_2)}{\text{inc}(\mathbf{P})}.$$

The *reversal ratio of* $\mathbf{P}$ is

$$RR(\mathbf{P}) = \frac{\text{led}(\mathbf{P})}{\text{inc}(\mathbf{P})} = \max_{L_1, L_2} RR(\mathbf{P}; L_1, L_2).$$

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  - Unpublished example difficult to analyze.

How small can $RR(\mathbf{P})$ be?

Theorem (BK 2011+)

For every sufficiently large positive integer k, there exists a poset $\mathbf{P}_k$ of width k with $RR(\mathbf{P}_k) \leq C/\log k$.

Doubling Property

Definition

Let $\mathbf{G} = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = k$. We say that $\mathbf{G}$ has the doubling property if for every $Y \subset A$ with $|Y| \leq k/3$, $|N(A)| \geq 2|A|$.

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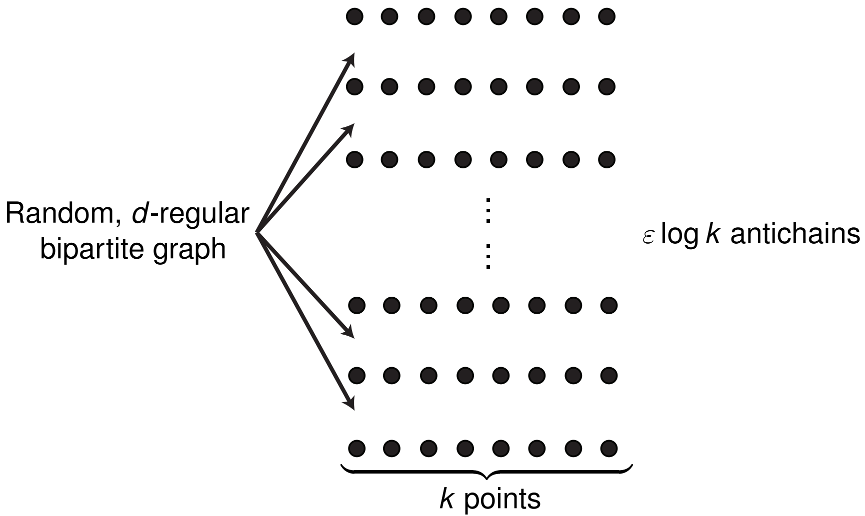
Lemma

Let $G_d(A, B)$ be a random d-regular bipartite graph on vertex sets A and B of size k, chosen according to the configuration model. For each $d \geq 10$ and k sufficiently large, $G_d(A, B)$ has the doubling property with high probability.

Construction of $\mathbf{P}_k$



Analysis of $\mathbf{P}_k$

Proposition

The number of incomparable pairs in $\mathbf{P}_k$ *is at least*

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Proposition

For k sufficiently large,

$$\operatorname{led}(\mathbf{P}_k) \leq \frac{19}{3}\varepsilon k^2 \log k.$$

Posets of fixed dimension

Definition

For $d \geq 2$, define $g_{RR}(d) = \inf\{RR(\mathbf{P})\colon \dim(\mathbf{P}) = d\}$.

Bounding $g_{RR}(d)$

- Standard example $\mathbf{S}_n$?

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- Standard example $\mathbf{S}_n$? $RR(\mathbf{S}_n) \to 1$
- d-dimensional grid? $RR(\mathbf{n}^d) \to 1/2$ (Felsner-Massow)
- $g_{RR}(3) = 2/3$ by considering $\mathbf{n}^3$
- $1/2 \leq g_{RR}(4) \leq 4/7$

Dimension of $\mathbf{P}_k$

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Corollary

There is a constant C such that $g_{RR}(d) \leq C/\log d$ for all $d \geq 2$.

Posets of fixed width

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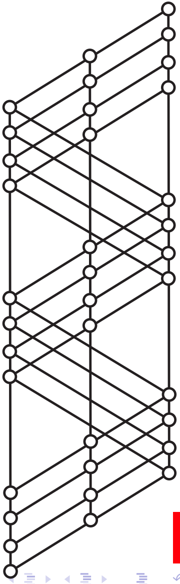
Proposition

$$h_{RR}(3) \leq 5/6$$

A vexing example

- $\mathbf{Z}_{t,k}$ has tk points per chain



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- $\mathbf{Z}_{t,k}$ has tk points per chain
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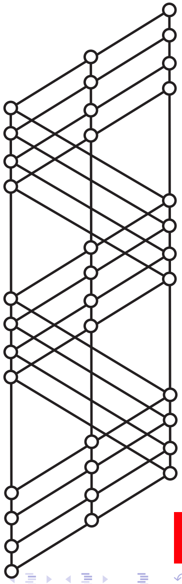

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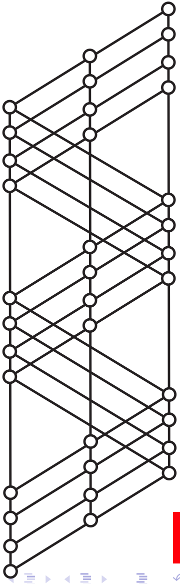

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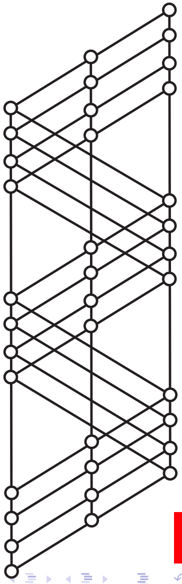
L_1	L_2
C_4	B_4/C_4
$C_3/A_4/B_4$	A_4
B_3	$A_3/B_3/C_3$
$B_2/C_2/A_3$	C_2
A_2	$C_1/A_2/B_2$
$A_1/B_1/C_1$	B_1
	A_1



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Conjecture (BK)

$$h_{RR}(3) = 3/4$$

Thank You

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Contact

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