MATH 3012B Final Exam

Fall 2008

Name: _____

GTid (9xxxxxxxx): _____

Group: _____

Instructor: Mitchel T. Keller

There are 15 questions on this exam on 3 pages (not counting this coverpage). **Answer questions 1 and 2 on a single solution sheet. Answer each of the remaining questions on separate solution sheets.** Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name, group, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Student signature: _____

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
15	5	
Total:	75	

1. (5 points) Let $X = \{m, a, t, h, r, u, l, e, s\}$.
   (a) How many X-strings of length 6 are there if repetition of symbols is allowed?
   (b) How many X-strings of length 6 are there if repetition of symbols is not allowed?
   (c) How many X-strings of length 6 are there if repetition of symbols is allowed but the string must contain exactly 2 m's?

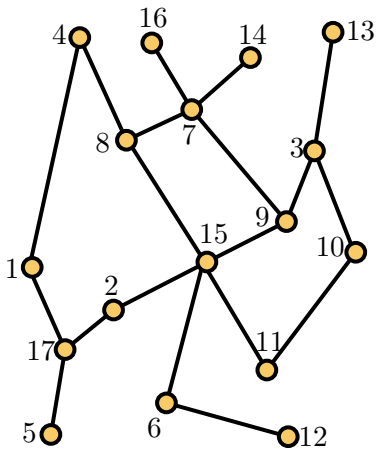
2. (5 points) How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 < 93$$

   if $x_1 \geq 10$, $x_2 > 0$, $x_3 \geq 0$, and $0 \leq x_4 \leq 5$?

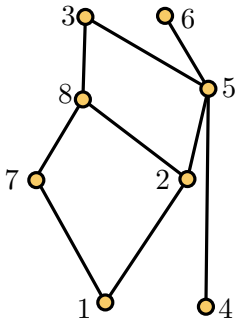
3. (5 points) Using the fact that a tree on n vertices has $n - 1$ edges, show that a tree on $n \geq 2$ vertices has at least two leaves.

4. (5 points) Below is a poset $\mathbf{P}$.
   (a) Find a maximal antichain of size three in $\mathbf{P}$.
   (b) Is the chain $\{17, 2, 8, 7, 16\}$ maximal? If it is, explain why. If it is not, list point(s) that should be added to make it maximal.
   (c) List the maximal elements of $\mathbf{P}$.

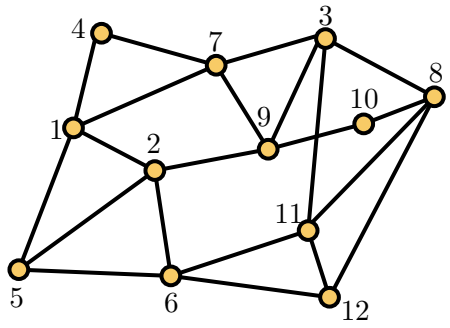


5. (5 points) Use the algorithm from the book/class to find the height h of the poset $\mathbf{P}$ in problem 4 and a partition of $\mathbf{P}$ into h antichains. Give the antichains by labeling the points in antichain i with i in the copy of the figure on the designated solution sheet.

6. (5 points) Explain why the poset below is not an interval order.



7. (5 points) Use the algorithm from the book/class to find an eulerian circuit in the graph below or explain why one doesn't exist. This graph is hamiltonian. List the vertices in the order of a hamiltonian cycle. Notice that the degree of vertex 4 is less than $6 = 12/2$. Explain why this does not contradict Dirac's Theorem.



8. (5 points) Alice and Bob are enjoying a relaxing day coloring graphs. Alice is having a hard time coloring a graph on 26 vertices. She found a proper coloring using five colors but says that because the graph's largest clique has only two vertices, its chromatic number should be 2. Bob says that since the graph has only 71 edges, it must be planar and so Alice should be able to use at most four colors. Their friend Christina drops by for a visit and claims that the chromatic number of the graph is in fact 5 and points out the problems in both of their arguments. What's wrong with Alice's and Bob's logic?

9. (5 points) Show that the number d_6 of derangements of $\{1, 2, 3, 4, 5, 6\}$ is given by

$$d_6 = \binom{6}{0}(6-0)! - \binom{6}{1}(6-1)! + \binom{6}{2}(6-2)! - \binom{6}{3}(6-3)! + \binom{6}{4}(6-4)! - \binom{6}{5}(6-5)! + \binom{6}{6}(6-6)!.$$

(You may not simply appeal to the formula for d_n proved in class. You must explain why this particular expression is true.)

10. (5 points) A candy company is preparing boxes of mixed candy canes to sell during the holiday season. They produce five flavors of candy canes: peppermint, cinnamon, chocolate, lemon, and cherry. How many ways can they prepare a box of n candy canes if the number of peppermint candy canes must be a multiple of four and each box must contain at most three cinnamon candy canes, at least one chocolate candy cane, zero or three lemon candy canes, and any number of cherry candy canes?

11. (5 points) Let p_n denote the number of permutations of $\{1, 2, \ldots, n\}$ in which each integer i is either in position i or adjacent to position i. Derive (but do not solve) a recurrence for p_n.

12. (5 points) Find the general solution of the recurrence equation

$$f_{n+2} = 3f_n - 2f_{n+1} + 3n2^n.$$

13. (5 points) Alice and Bob are discussing friends to invite to their party to celebrate Georgia Tech's 45-42 victory over U[sic]GA. Bob says that if six friends attend, then among the six guests (Alice and Bob excluded), they will be guaranteed that, regardless of who those friends are, there will be either a group of four people each of whom has met all the others in the group of four or a group of four people none of whom has met any of the others in the group of four. Alice disagrees that six is enough to accomplish this but says she's sure 20 guests would be enough. Who's right and why is he/she correct and the other incorrect?

14. (5 points) Students are preparing to do final projects for an applied combinatorics course. The five possible topics for their final projects are Ramsey theory, posets, induction, graph theory, and generating functions. There are five students in the class, and they have each given their professor the list of topics on which they are willing to do their project. Ahmed is interested in posets or graphs. Bianca would be willing to do her project on Ramsey theory, posets, or induction. Calvin will only consider posets or graphs. Dexter likes generating functions and induction. Ezmerelda wants to do her project on either graphs or posets. To prevent unauthorized collaboration, the professor does not want to have two students work on the same topic. Is it possible to assign each student a topic from the lists above so that no two students work on the same project? If so, find such an assignment. If not, find an assignment that maximizes the number of students who have assignments from their lists and explain why you cannot satisfy all the students' requests.

15. (5 points) Below is a poset $\mathbf{P}$ and a network used to find a chain partition of the poset. (All edges in the network have a capacity of 1 and the **bold** edges currently carry a flow of 1.) Using the network, find the width w of $\mathbf{P}$, a partition of $\mathbf{P}$ into w chains, and an antichain with w elements. (Answer this problem on the designated solution sheet.)