1. (5 points) The setting for this problem is the fictional town of Mascotville, which is laid out as a grid. Mascots are allowed to travel only on the streets, and not "as the yellowjacket flies." Buzz wants to go visit his friend Thundar, the North Dakota State University mascot, who lives 6 blocks east and 7 blocks north of Buzz's hive. However, Uga VII has recently moved into the doghouse 2 blocks east and 3 blocks north of Buzz's hive and already has a restraining order against Buzz. There's also a pair of tigers from Clemson who live 1 block east and 2 blocks north of Uga VII, and they're known for setting traps for Buzz. Buzz wants to travel from his hive to Thundar's pen every day without encountering Uga VII or The Tiger and The Tiger Cub, but wants to avoid the boredom caused by using a route he's used in the past. What is the largest number of consecutive days on which Buzz can make the trip to visit Thundar without reusing a route?

Solution: At its core, this is a lattice path problem. Fix the coordinates of Buzz's hive as $(0, 0)$. Then Thundar's pen is at $(6, 7)$, Uga VII's doghouse is at $(2, 3)$, and the tigers are at $(3, 5)$. Thus, the question is the number of lattice paths from $(0, 0)$ to $(6, 7)$ that do not pass through $(2, 3)$ or $(3, 5)$, since after that many days, Buzz will be forced to reuse a route.

The total number of lattice paths from $(0, 0)$ to $(6, 7)$ is $T := C(13, 6)$. Of these, $U := C(5, 2)C(8, 4)$ pass through $(2, 3)$ and $C := C(8, 3)C(5, 3)$ pass through $(3, 5)$. However, there are some lattice paths that pass through both $(2, 3)$ and $(3, 5)$, so if we just subtracted U and C from T, we'd have removed those lattice paths twice. Thus, we need to add back the lattice paths passing through both $(2, 3)$ and $(3, 5)$. There are $B := C(5, 2)C(3, 1)C(5, 3)$ of these, giving

$$T - U - C + B = \binom{13}{6} - \binom{5}{2}\binom{8}{4} - \binom{8}{3}\binom{5}{3} + \binom{5}{2}\binom{3}{1}\binom{5}{3} = 756$$

different routes that Buzz can take to visit Thundar while avoiding Uga VII and The Tiger and The Tiger Cub, meaning he can make the trip every day for over two years without getting bored.