1. (5 points) If $\mathbf{G} = (V, E)$ is a graph, its *degree sequence* is the list of the degrees of its vertices. That is, if $V = \{v_1, v_2, \ldots, v_n\}$, the degree sequence of $\mathbf{G}$ is $(\deg_{\mathbf{G}}(v_1), \deg_{\mathbf{G}}(v_2), \ldots, \deg_{\mathbf{G}}(v_n))$. (By convention, we assume that the vertices are arranged such that $\deg_{\mathbf{G}}(v_1) \geq \deg_{\mathbf{G}}(v_2) \geq \cdots \geq \deg_{\mathbf{G}}(v_n)$.) Below are five sequences of integers (along with n, the number of integers in the sequence). Identify

   - the *one* sequence that **cannot be the degree sequence of any graph**;
   - the *two* sequences that could be the degree sequence of a **planar** graph;
   - the *one* sequence that could be the degree sequence of a **tree**;
   - the *one* sequence that is the degree sequence of an **eulerian** graph; and
   - the *one* sequence that is the degree sequence of a graph that must be **hamiltonian**.

   Be sure to explain your answers. (Note that one sequence will get two labels from above.)

   (a) $n = 10$: $(4, 4, 2, 2, 1, 1, 1, 1, 1, 1)$

   Solution: This is the only degree sequence that could be a **tree**, as it is the only one that has at least two vertices of degree one, and we know that a tree has at least two leaves. It is also one of the two that can be **planar**, since it has $18/2 = 9 < 3 \cdot 10 - 6 = 24$ edges.

   (b) $n = 9$: $(8, 8, 8, 6, 4, 4, 4, 4, 4)$

   Solution: This is the only sequence in which all degrees are even, so it is the **eulerian** graph. It cannot be planar since it has $50/2 = 25 > 3 \cdot 9 - 6 = 21$ edges.

   (c) $n = 7$: $(5, 4, 4, 3, 2, 1, 0)$

   Solution: Such a graph would have three vertices of odd degree, and since a graph cannot have an odd number of vertices of odd degree, there is **no such graph**.

   (d) $n = 10$: $(7, 7, 6, 6, 6, 6, 5, 5, 5, 5)$

   Solution: This graph cannot be planar, since it would have $58/2 = 29 > 3 \cdot 10 - 6 = 24$ edges. However, since each vertex has degree at least $n/2 = 5$, the graph is **hamiltonian** by Dirac's Theorem.

   (e) $n = 6$: $(5, 4, 3, 2, 2, 2)$

   Solution: This graph is not eulerian, since it has vertices of odd degree. It has no vertices of degree one, so it is not a tree. It has vertices of degree $2 < 3 = 6/2$, so it need not be hamiltonian. However, it has $9 < 3 \cdot 6 - 6 = 12$ edges, so it could be the degree sequence of a **planar** graph.