MATH 3012B Test I

Fall 2008

Name: ______________________________

GTid (9xxxxxxxx): ______________________________

Instructor: Mitchel T. Keller

There are 11 questions on this exam on 7 pages (not counting this coverpage). Answer questions **in order** on the provided solution sheets. Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name and problem number(s) on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, twelve points will be deducted from your raw total score on this exam.**

Student signature: ______________________________

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	5	5	5	5	5	5	5	5	5	5	5	55
Score:												

In questions 1 and 2, we consider strings whose characters are taken from the 26-letter (lower-case) English alphabet and the set of 10 arabic digits. Let X be this set of characters.

1. (5 points) (See above for definition of X.)

   (a) How many X-strings of length 10 are there?

   Solution: Since $|X| = 36$ and the strings are to have length 10, we have that there are 36^{10} X-strings of length 10.

   (b) How many X-strings of length 10 are there if repetition of symbols is not allowed.

   Solution: If repetition is not allowed, we're looking for a permutation of length 10 from a set of 36 elements, so the number is $P(36, 10) = 36!/26!$.

2. (5 points) (See above for definition of X.) How many X-strings α of length 15 satisfy **all** of the following properties (at the same time): (i) the first and last symbols of α are distinct digits (which may appear elsewhere in α), (ii) precisely four of the symbols in α are the letter 't', and (iii) precisely three elements of the set $V = \{a, e, i, o, u\}$ appear in α?

   Solution: We can fill in the first and last symbols of α in $P(10, 2) = 10 \cdot 9$ ways. Having done this, there are $C(13, 4)$ ways to choose where the four t's appear. With this done, we must choose 3 of the remaining 9 spaces to fill with elements of V; this can be done in $C(9, 3)$ ways. There are then $P(5, 3)$ ways to fill in those three spots. With all this said and done, there are six spots left to fill, and they can be filled with any of 30 symbols (any digit or any letter not in $V \cup \{t\}$, which has size 6), giving 30^6 ways to fill these spots. Thus, there are

   $$P(10,2)\binom{13}{4}\binom{9}{3}P(5,3)30^6$$

   strings of the type described in this question.

3. (5 points) What is the coefficient on $x^{10}y^{74}$ in the expansion of $(3x^5 + x^2y + y + 1)^{100}$?

   Solution: There are two ways to get an $x^{10}y^{74}$ term when expanding $(3x^5 + x^2y + y + 1)^{100}$. The first is to raise $3x^5$ to the 2^{nd} power, y to the 74^{th} power, x^2y to the 0^{th} power, and 1 to the 24^{th} power. The second is to raise $3x^5$ to the 0^{th} power, x^2y to the 5^{th} power, y to the 69^{th} power, and 1 to the 26^{th} power. Put another way, the multinomial theorem says that

   $$(3x^5 + x^2y + y + 1)^{100} = \sum_{k_1+k_2+k_3+k_4=100} \binom{100}{k_1, k_2, k_3, k_4} (3x^5)^{k_1} (x^2y)^{k_2} y^{k_3} 1^{k_4},$$

and the two ways to get and $x^{10}y^{74}$ term are (i) $k_1 = 2$, $k_2 = 0$, $k_3 = 74$, and $k_4 = 24$ and (ii) $k_1 = 0$, $k_2 = 5$, $k_3 = 69$, and $k_4 = 26$. In case (i), we get a coefficient of

$$\binom{100}{2, 0, 74, 24} 3^2$$

and in (ii) we get

$$\binom{100}{0, 5, 69, 26}.$$

Thus, overall, we combine like terms and have that the coefficient on $x^{10}y^{74}$ is

$$\binom{100}{2, 0, 74, 24} 3^2 + \binom{100}{0, 5, 69, 26}.$$

4. (5 points) How many integer solutions are there to the inequality

$$y_1 + y_2 + y_3 + y_4 < 181$$

with $y_1, y_2 \geq 0$, $y_3 > 4$, and $y_4 \geq 10$? What if we add the restriction that $y_3 \leq 50$?

Solution: We think of adding a fifth variable to take up the slack, and this variable must be positive in order to account for the strict inequality we are given. We start by allocating 4 objects to y_3 and 10 to y_4, leaving us with 167 objects to distribute. Three of our "employees" can now get zero objects, meaning we need three artificial objects, upping the count to 170. These determine 169 gaps, and we must choose four of them (to divide into five pieces). This means there are

$$\binom{169}{4}$$

integer solutions to the given inequality.

When the restriction $y_3 \leq 50$ is added, we can handle this by finding the number of solutions in which $y_3 > 50$ and subtracting it from the total number of solutions found above. In this case, instead of allocating 4 objects to y_3 initially, we allocate 50, leaving 121. Again, we need three artificial objects, giving 124. We can choose 4 of the 123 gaps in $C(123, 4)$ ways. Thus, with the added constraint, we have

$$\binom{169}{4} - \binom{123}{4}$$

integer solutions.

5. (5 points) Find $d := \gcd(413, 107)$ and integers a, b such that $413a + 107b = d$.

Solution: In the table below, the left column uses the division theorem, and then the right column shows how this allows us to simplify the greatest common divisor computation via the Euclidean algorithm.

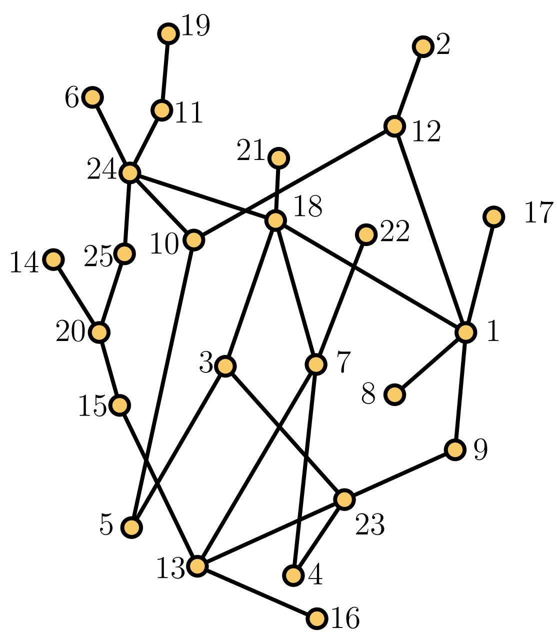
$413 = 3 \cdot 107 + 92$	$\gcd(413, 107)$	$= \gcd(107, 92)$
$107 = 1 \cdot 92 + 15$		$= \gcd(92, 15)$
$92 = 6 \cdot 15 + 2$		$= \gcd(15, 2)$
$15 = 7 \cdot 2 + 1$		$= \gcd(2, 1) = 1$

Thus, $\gcd(413, 107) = 1$. Solving the equations in the left column above for the remainder and working backward, we have

$$\begin{aligned} 1 &= 15 - 7 \cdot 2 \\ &= 15 - 7 \cdot (92 - 6 \cdot 15) = 43 \cdot 15 - 7 \cdot 92 \\ &= 43 \cdot (107 - 1 \cdot 92) - 7 \cdot 92 = 43 \cdot 107 - 50 \cdot 92 \\ &= 43 \cdot 107 - 50(413 - 3 \cdot 107) \\ &= 193 \cdot 107 - 50 \cdot 413 \end{aligned}$$

Thus, $a = -50$ and $b = 193$ satisfy $413a + 107b = 1$.

6. (5 points) Consider the poset $\mathbf{P}$ shown below:



(a) List the set of maximal elements of **P**.

> **Solution:** The maximal elements of **P** are $14, 6, 19, 2, 21, 22$, and 17.

(b) List the set of elements of **P** that are incomparable to 24.

> **Solution:** The set of elements of **P** that are incomparable to point 24 is $\{14, 21, 2, 12, 17, 22\}$.

(c) How many elements of $\{2, 3, \ldots, 25\}$ are comparable to 1?

> **Solution:** There are 14 elements ($\{17, 9, 23, 4, 13, 16, 18, 21, 24, 6, 11, 19, 12, 2\}$) of **P** comparable to 1.

7. (5 points) Consider again the poset **P** from question 6.

(a) Writing directly on the figure on the solution sheet, carry out the algorithm to partition **P** into the least possible number of antichains. Label the first antichain with 1, the second with 2, etc.

> **Solution:** See the scanned solution sheet at the end of this document.

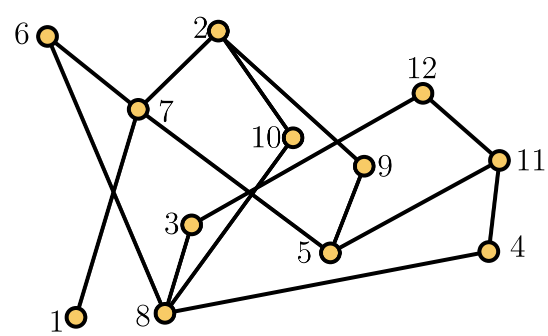
(b) Find the height h of **P**.

> **Solution:** Since the largest label we used in the previous part was 9, height$(\mathbf{P}) = 9$.

(c) List a set of h elements that forms a chain.

> **Solution:** Starting with the point labelled 9 and working down, we find that $\{19, 11, 24, 18, 1, 9, 23, 13, 16\}$ is a set of 9 elements that forms a chain.

8. (5 points) Consider the poset **Q** shown below:



(a) Find the width w of $\mathbf{Q}$.

> **Solution:** The width of $\mathbf{Q}$ is 5, as the following parts show.

(b) Find an antichain in $\mathbf{Q}$ of size w.

> **Solution:** The set $\{7, 3, 10, 9, 11\}$ is an antichain of size 5.

(c) Find a chain partition of $\mathbf{Q}$ into w chains. Give your answer by labelling the points of $\mathbf{Q}$ (using the copy on your solution sheet) with elements of $\{1, 2, \ldots, w\}$ so that for each i, the elements labelled i form a chain.

> **Solution:** See the scanned solution sheet at the end of this document.

9. (5 points) Let x and y be positive integers. Give a combinatorial proof (i.e., explain how both sides of the equation count the same thing, but in different ways) of the fact that for integers $n \geq 0$

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

> **Solution:** The right-hand side counts the number of ways to choose n elements from an $x + y$-element set. We can see that the left-hand side counts this same number of things as well. However, it first partitions the set of $x + y$ elements into a set of x elements and a set of y elements. Each of the ways of choosing n elements from the $x + y$-element set results in some number k of elements of the x-element set being chosen, where $0 \leq k \leq n$. The other $n - k$ elements must come from the y-element set. For fixed k, there are thus $C(x, k)C(y, n - k)$ ways to choose n elements in this manner. Summing over all possible values of k accounts for all possible n-element subsets, and this is precisely the left-hand side.

10. (5 points) Suppose that we extend our definition of lattice paths to allow not only horizontal moves (increasing the first coordinate by 1 and the second by 0) and vertical moves (increasing the first coordinate by 0 and the second by 1) but also *diagonal moves* in which *both* coordinates are increased by 1. Call such a path a D-lattice path, since we allow diagonal moves, denoted by D. Let $f(m, n)$ denote the number of D-lattice paths from $(0, 0)$ to (m, n). Show that $f(m, n)$ satisfies the recurrence

$$f(m, n) = f(m - 1, n) + f(m, n - 1) + f(m - 1, n - 1)$$

for $m, n \geq 1$. Use this recurrence to find $f(1, 3)$.

Solution: A D-lattice path can be thought of as a string of the symbols D, H, and V, standing for diagonal, horizontal, and vertical moves, respectively. We can partition all D-lattice paths from $(0,0)$ to (m,n) into three parts, depending on whether the last move is an H, a V, or a D. In the first case, removing the H leaves a D-lattice path from $(0,0)$ to $(m-1,n)$, and there are $f(m-1,n)$ of these. In the second case, removing the V leaves a D-lattice path from $(0,0)$ to $(m,n-1)$, and there are $f(m,n-1)$ of these. In the final case, removing the D results in a D-lattice path from $(0,0)$ to $(m-1,n-1)$, and there are $f(m-1,n-1)$ of these. Thus, the total number of D-lattice paths from $(0,0)$ to (m,n) is

$$f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1).$$

Notice that $f(0,n) = f(m,0) = 1$, since if either coordinate is zero, there's only one D-lattice path that can be used, which consists of either all vertical or all horizontal moves.

Now we compute

$$\begin{aligned}
f(1,3) &= f(0,3) + f(1,2) + f(0,2) \\
&= 2 + f(1,2) \\
&= 2 + f(0,2) + f(1,1) + f(0,1) \\
&= 4 + f(1,1) \\
&= 4 + f(0,1) + f(1,0) + f(0,0) \\
&= 7.
\end{aligned}$$

11. (5 points) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$. Prove that $f(1) = 0$ and that $f(u^n) = nu^{n-1}f(u)$ for all integers $n \geq 0$ and $u \in \mathbb{R}$. (*Hint*: To show $f(1) = 0$, note that $f(1) = f(1 \cdot 1)$.)

Solution: We first show that $f(1) = 0$. From the hint and the given property of $f(xy)$, we have

$$f(1) = f(1 \cdot 1) = 1 \cdot f(1) + 1 \cdot f(1) = 2f(1).$$

Solving this for $f(1)$ shows that $f(1) = 0$.

We prove the remainder by induction. The basis step is for $n = 0$, which leaves us to show $f(u^0) = 0u^{-1}f(u)$. But now the left-hand side of this is $f(1)$ and the right-hand side is 0, and we know that $f(1) = 0$, so the basis step is complete.

Now suppose that for some $k \geq 0$ we have $f(u^k) = ku^{k-1}f(u)$ for all $u \in \mathbb{R}$. Consider $f(u^{k+1})$. We have

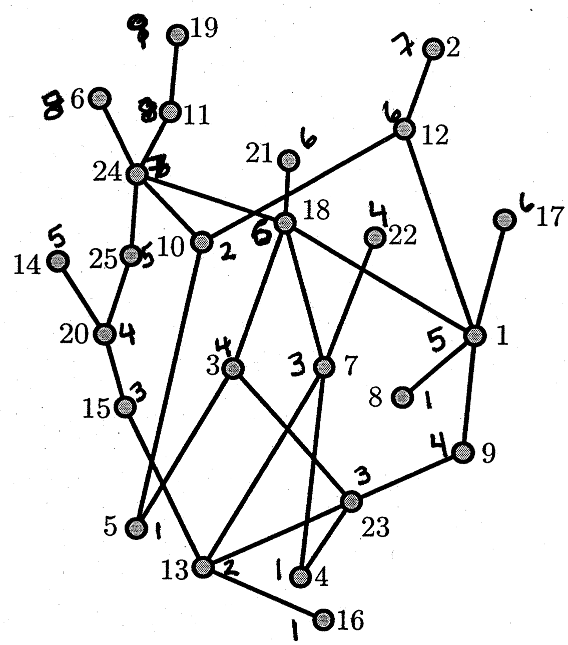
$$f(u^{k+1}) = f(u \cdot u^k) = uf(u^k) + u^k f(u).$$

Now by the inductive hypothesis, $f(u^k) = ku^{k-1}f(u)$, so substituting this gives

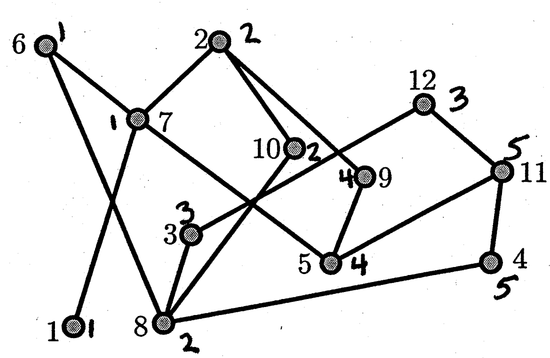
$$f(u^{k+1}) = u \cdot ku^{k-1}f(u) + u^k f(u) = ku^k f(u) + u^k f(u) = (k+1)u^k f(u).$$

This is what we needed to show, so by the Principle of Mathematical induction, we have $f(u^n) = nu^{n-1}f(u)$ for all integers $n \geq 0$ and all $u \in \mathbb{R}$.

Use this figure and the space following for question 7:



Use this figure and the space following for question 8:



Use back for scratchwork. Back will **not** be graded. Use the front of an additional sheet if you need more space.