MATH 3012B Test II

Fall 2008

Name: SOLUTIONS

GTid (9xxxxxxxx): ____________________

Instructor: Mitchel T. Keller

There are 10 questions on this exam on 2 pages (not counting this coverpage). Answer questions **in order** on the provided solution sheets. Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name and problem number(s) on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

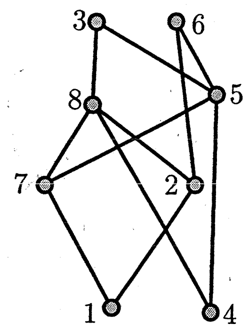
I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, twelve points will be deducted from your raw total score on this exam.**

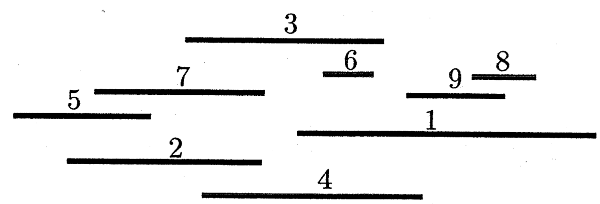
Student signature: ____________________

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	5	5	5	5	5	5	5	5	5	5	50
Score:											

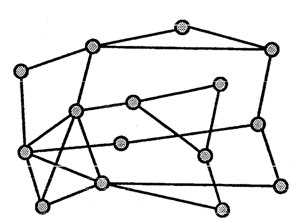
1. (5 points) Determine if the poset below is an interval order. If it is not, explain why not. If it is, use the algorithm from the book/class to find an interval representation for it.



2. (5 points) Below is a collection of intervals that give the interval representation of an interval order. Apply the First Fit algorithm to find the width of this interval order and a partition into as few chains as possible. (Show your chain partition by labeling the copy of the figure on the designated solution sheet. You are not required to answer this question with a complete sentence.) Find a maximum antichain in this interval order.



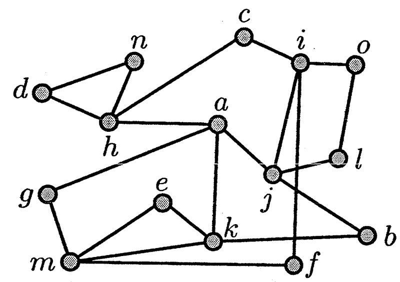
3. (5 points) Is the graph **G** shown below planar? If not, explain why not. If it is, give a planar drawing. Find a proper coloring of **G** using the smallest number of colors possible and explain why you cannot use fewer colors. (Give the coloring by labeling the vertices on the copy of the graph on the designated solution sheet.)



4. (5 points) How many (labeled) graphs are there with vertex set $\{1, 2, \ldots, n\}$? (We're using "labeled" here just like we did with trees. We don't care if the graphs are isomorphic in this case; they are the same only if they have the same edge set.) How many (labeled) graphs with vertex set $\{1, 2, \ldots, n\}$ have exactly k edges?

5. (5 points) Alice and Bob are discussing Dirac's Theorem about hamiltonian cycles in graphs. Alice claims that this theorem implies every graph on 10 vertices with 37 edges must be hamiltonian, since $37 > 25 = ((10/2) \cdot 10)/2$. Bob says that Alice doesn't understand the theorem. Which one is right and why?

6. (5 points) Using the algorithm from the book/class, find an eulerian circuit in the graph below. Your answer need not include a complete sentence of explanation, but must show the partial circuits you find at each step.



7. (5 points) Bob has been studying planar graphs and graph coloring. He claims that he can properly color any graph on 18 vertices that has exactly 45 edges using at most four colors. Alice tells Bob that he should hit the books some more, because she can draw a graph on 18 vertices with 45 edges that has chromatic number 10. Who is correct, and why?

8. (5 points) Find and draw the labelled tree with Prüfer code 31173. Your answer need not include a complete sentence of explanation if you use the algorithm for doing this from class/the book.

9. (5 points) Recall that a function $f\colon X \to Y$ is a *surjection* or *onto* if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Use the Principle of Inclusion-Exclusion to find the number of surjections from $\{1, 2, 3, 4, 5, 6, 7\}$ to $\{1, 2, 3, 4\}$.

10. (5 points) Given that $333\,499\,353 = 3 \cdot 11^3 \cdot 17^4$, compute $\phi(333\,499\,353)$.

(1)

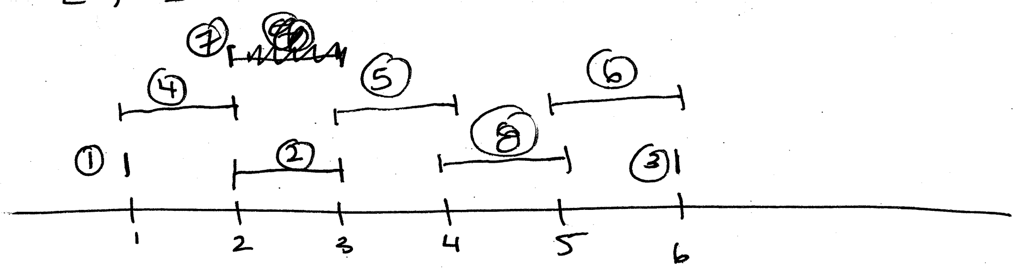
$D(1) = \{\}$
$D(2) = \{1\}$
$D(3) = \{1,2,4,5,7,8\}$
$D(4) = \{\}$
$D(5) = \{1,4,7\}$
$D(6) = \{1,2,4,5,7\}$
$D(7) = \{1\}$
$D(8) = \{1,2,4,7\}$

$U(1) = \{2,3,5,6,7,8\}$
$U(2) = \{3,6,8\}$
$U(3) = \{\}$
$U(4) = \{3,5,6,8\}$
$U(5) = \{3,6\}$
$U(6) = \{\}$
$U(7) = \{3,5,6,8\}$
$U(8) = \{3\}$

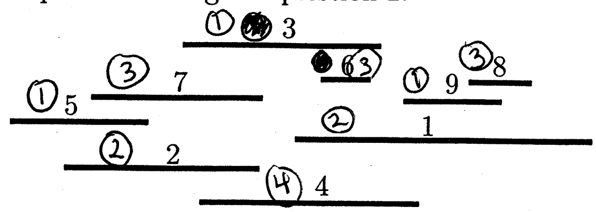
$$\underset{1,\ 4}{\overset{D_1}{\{\}}} \subseteq \underset{2,\ 7}{\overset{D_2}{\{1\}}} \subseteq \underset{5}{\overset{D_3}{\{1,4,7\}}} \subseteq \underset{8}{\overset{D_4}{\{1,2,4,7\}}} \subseteq \underset{6}{\overset{D_5}{\{1,2,4,5,7\}}} \subseteq \underset{3}{\overset{D_6}{\{1,2,4,5,7,8\}}}$$

$$\underset{1}{\overset{u_1}{\{2,3,5,6,7,8\}}} \supseteq \underset{4,\ 7}{\overset{u_2}{\{3,5,6,8\}}} \supseteq \underset{2}{\overset{u_3}{\{3,6,8\}}} \supseteq \underset{5}{\overset{u_4}{\{3,6\}}} \supseteq \underset{8}{\overset{u_5}{\{3\}}} \supseteq \underset{3,\ 6}{\overset{u_6}{\{\}}}$$

$I(1) = [1,1]$
$I(2) = [2,3]$
$I(3) = [6,6]$
$I(4) = [1,2]$
$I(5) = [3,4]$
$I(6) = [5,6]$
$I(7) = [2,3]$
$I(8) = [4,5]$



Use this figure and the space following for question 2:

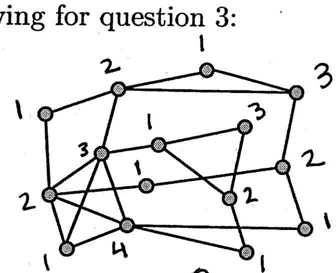


width = 4

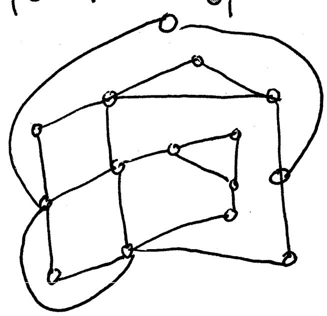
4-element (maximum) antichain: {3, 7, 2, 4}

Use this figure and the space following for question 3:

Proper 4-coloring:



Planar:



Chromatic number is 4, since the Four Color Theorem guarantees that planar graphs have chromatic number at most four and there is a 4-clique, which requires four colors to properly color.

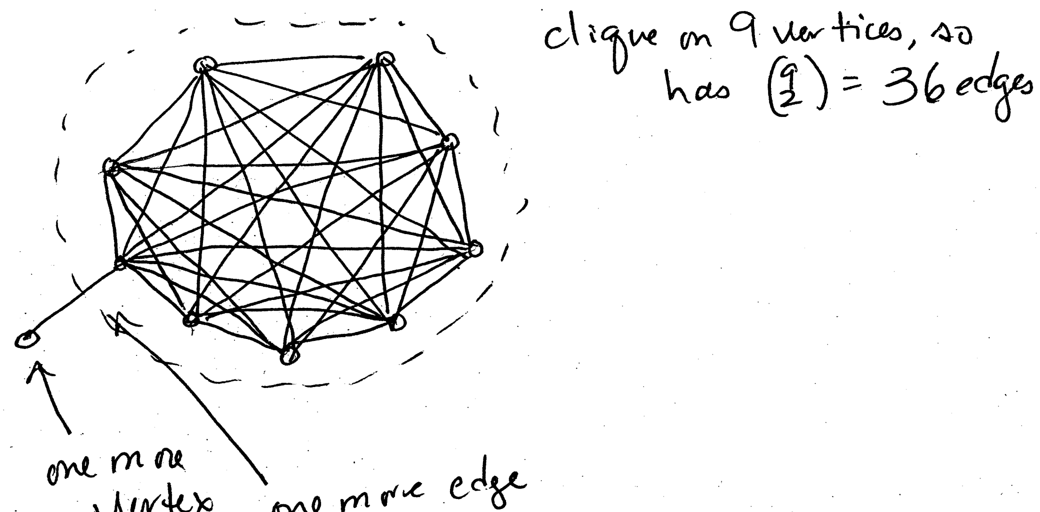
(4) An edge is a 2-element subset of the vertex set, in this case $[n]$. Let $\mathcal{E}$ denote the set of all $\binom{n}{2}$ 2-element subsets of $[n]$. A graph with vertex set $[n]$ is thus determined simply by its edge set, which is a subset of $\mathcal{E}$. There are $2^{\binom{n}{2}}$ subsets of $\mathcal{E}$ since $|\mathcal{E}| = \binom{n}{2}$, so there are $2^{\binom{n}{2}}$ graphs with vertex set $[n]$.

(Another way to look at it: for each of the $\binom{n}{2}$ possible edges, you can either have it in your graph or not have it. Two choices per edge, so $2^{\binom{n}{2}}$ graphs.)

To have exactly k edges, we choose k elements of $\mathcal{E}$, and each of the $\binom{|\mathcal{E}|}{k}$ subsets of $\mathcal{E}$ with k elements gives a graph with exactly k edges and vertex set $[n]$. Thus there are

$$\binom{|\mathcal{E}|}{k} = \binom{\binom{n}{2}}{k}$$ such graph.

(5) Bob is correct. Dirac's Theorem says that a graph on n vertices in which every vertex has degree at least $\lceil \frac{n}{2} \rceil$ is hamiltonian. It is possible for a graph on 10 vertices to have 37 edges but also have a vertex with degree less than $\lceil \frac{n}{2} \rceil$, and thus Dirac's Theorem does not apply. As an example:



This graph is not hamiltonian, since it has a vertex of degree one.

(6) agmeka

ahcifmkbja Expands to ahcifmkbjagmeka

hdnh

Expands to ahdnhcifmkbjagmeka

ijloi

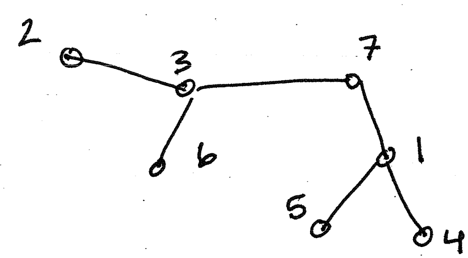
Expands to ahdnhcijloifmkbjagmeka

(7) Alice is correct. As an example, consider the 18-vertex graph consisting of a 10-vertex clique and 8 vertices of degree zero. This graph has chromatic number 10, 18 vertices, and 45 edges.

Bob's mistake is thinking that any graph on 18 vertices with less than or equal to $3 \cdot 18 - 6 = 48$ edges is planar, which our example shows is not the case.

(8)

Prüfer code	Label list	Edge added
31173	1,2,3,4,5,6,7	2-3
1173	1,3,4,5,6,7	4-1
173	1,3,5,6,7	5-1
73	1,3,6,7	1-7
3	3,6,7	6-3
empty string	3,7	3-7



(9) For $i = 1, 2, 3, 4$, define property P_i to be i is not in the image of a function $f: [7] \to [4]$. (That is, there is no $x \in [7]$ s.t. $f(x) = i$.)

The number of surjections ~~is~~ from $[7]$ to $[4]$ is thus the number of functions from $[7]$ to $[4]$ satisfying none of the P_i, so we may apply the principle of Inclusion-Exclusion.

Note that $N(S)$, for $S \subseteq [4]$, depends only on the size of S. If ~~so~~ $|S| = k$, then there are $(4-k)^7$ functions from $[7]$ to $[4]$ "missing" the k things in the set S. (Effectively, we've reduced the size of the range to be $4-k$, and we get to choose one of those $4-k$ values for each of 7 inputs.) Now we note that there are $\binom{4}{k}$ subsets of $[4]$ of size k, so we get $\binom{4}{k}$ terms of the form $(4-k)^7$. Thus, applying the Principle of Inclusion-Exclusion we have that the number of surjections from $[7]$ to $[4]$ is

$$\binom{4}{0} 4^7 - \binom{4}{1}(4-1)^7 + \binom{4}{2}(4-2)^7 - \binom{4}{3}(4-3)^7 + \binom{4}{4}(4-4)^7$$

$$= 4^7 - 4 \cdot 3^7 + 6 \cdot 2^7 - 4 \cdot 1^7$$

10

$$\varphi(3 \cdot 11^3 \cdot 17^4) = 3 \cdot 11^3 \cdot 17^4 \cdot \frac{3-1}{3} \cdot \frac{11-1}{11} \cdot \frac{17-1}{17}$$

$$= 3 \cdot 11^3 \cdot 17^4 \cdot \frac{2}{3} \cdot \frac{10}{11} \cdot \frac{16}{17}$$

$$= 2^6 \cdot 5 \cdot 11^2 \cdot 17^3$$