MATH 3012B Test III

Fall 2008

Name: ____

GTid (9xxxxxxxx): ______________________________________

Instructor: Mitchel T. Keller

There are 9 questions on this exam on 3 pages (not counting this coverpage). Answer questions **in order** on the provided solution sheets. Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name and problem number(s) on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, twelve points will be deducted from your raw total score on this exam.**

Student signature: ______________________________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	5	5	5	5	5	5	5	5	5	45
Score:										

1. (5 points) Find the generating function for the number a_n of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

subject to the constraints $x_1 \geq 3$, $0 \leq x_2 \leq 1$, x_3 is even, and x_4 is either 0, 2, or 6. Use your generating function to find a formula for a_n for $n \geq 8$ as well as the number of solutions for $n = 15$.

2. (5 points) Give the generating function $D(x)$ for the number of partitions of an integer n into distinct parts as well as the generating function $O(x)$ for the number of partitions of an integer n into odd parts. Use these generating functions to show that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

3. (5 points) Give the general solution of the advancement operator equation

$$(A-3)^3(A+2)(A-1)^2(A+6)f = 0.$$

4. (5 points) The n-cube $\mathbf{Q}_n$ is the graph whose vertex set is the set of all binary strings of length n. A string s is adjacent to another string s' in $\mathbf{Q}_n$ when they differ in exactly one place. (For instance, in $\mathbf{Q}_5$, the vertex 00110 is adjacent to the vertex 10110 but not to the vertex 11110.) Let q_n be the number of edges in $\mathbf{Q}_n$, so that $q_1 = 1$ and $q_2 = 4$. Derive a recurrence for q_n and use it to show that $q_n = n2^{n-1}$.

5. (5 points) Find a particular solution of the non-homogeneous advancement operator equation

$$(A-2)(A-3)f = 2n + 4 \cdot 2^n.$$

6. (5 points) Newly-named Wal-Mart CEO and Georgia Tech alumnus Mike Duke is touring the Athens, Georgia, Wal-Mart and makes a visit to the receiving dock to meet with low-level employees. There he meets Matthew S. and Knowshon M., who are having a heated debate about if the shipping routes used to get merchandise from the Bentonville, Arkansas, headquarters to Athens are being used optimally. All merchandise is shipped on trucks, but weight restrictions on the roads the trucks will travel over limit the amount of merchandise that can travel on any given route during a single day. Due to union requirements, the truck drivers must stop at intermediate warehouses where merchandise is unloaded and then placed on other trucks to continue the journey. Matthew S. argues that the largest amount of merchandise (by weight) is reaching the Athens store because he thought really hard and couldn't find a better way to do it. Knowshon M. looks at only the roads on the map leading directly to the Athens store and says they could receive 20 tons more merchandise each day. Mr. Duke quickly realizes (unsurprisingly) that they are both wrong and demonstrates a way to increase the amount of merchandise arriving in Athens each day and that the increase could not be as large as Knowshon M. suggests. How did he do this using material he learned in MATH 3012?

7. (5 points) The data file for a graph with vertex set $\{1, 2, \ldots, 7\}$ is shown below. (The first line gives the number of vertices, then each subsequent line gives the endpoints of the edge and the edge's weight.) On a solution sheet, list in order the edges of a minimum weight spanning tree that would be found in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm (build tree). **Vertex 1 is the root.** Be sure to clearly label the algorithm used for each list.

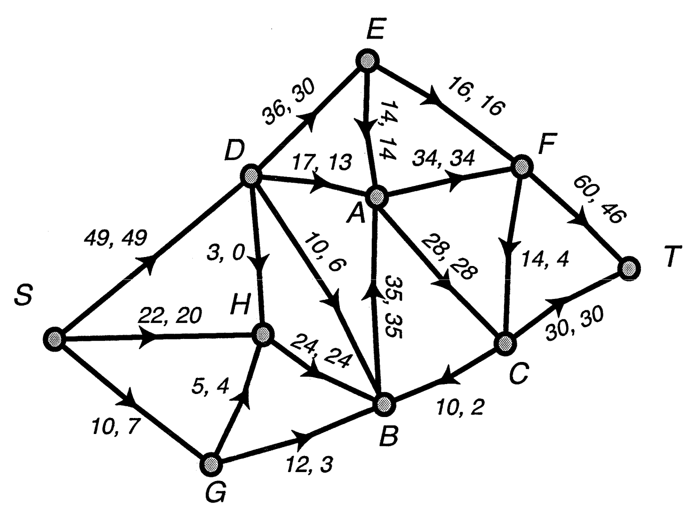
```
graph1.txt
7
1 5 34
5 3 12
7 5 43
2 3 20
4 7 41
3 6 52
2 5 17
4 2 16
1 4 38
6 7 51
```

8. (5 points) The matrix given below is the distance matrix for a digraph **D** whose vertex set is $\{A, B, C, D, E, F\}$. Apply Dijkstra's algorithm to find all the shortest paths from vertex A to all other vertices in the digraph **D**.

	A	B	C	D	E	F
A	0	7	17	55	83	42
B	14	0	13	47	27	17
C	37	42	0	16	93	28
D	10	6	8	0	4	32
E	84	19	42	8	0	45
F	36	3	76	5	17	0

(To be clear on how to read the matrix, the length of the edge from A to C is 17, while the length of the edge from C to A is 37.)

9. (5 points) Below is a network along with a flow (edges are labeled first with the capacity, then the current flow value). Find a maximum flow and minimum cut in this network using the Ford-Fulkerson Labeling Algorithm and show this flow by updating the amount of flow on edges on the diagram on the designated solution sheet. What is the value of the maximum flow? What is the capacity of the minimum cut?



Solution for problem(s) 1 MATH 3012B Name: SOLUTION

① Since $x_1 \geq 3$, the factor in the generating function that corresponds to x_1 is $x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x}$.

Since x_2 is either 0 or 1, its factor is $1+x$.

Since x_3 is even, its factor is $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$.

Since x_4 is either 0, 2, or 6, its factor is $1 + x^2 + x^6$.

Thus, our generating function is

$$\frac{x^3}{1-x} \cdot 1+x \cdot \frac{1}{1-x^2} \cdot (1+x^2+x^6) = \frac{x^3+x^5+x^9}{(1-x)^2}$$

$$= (x^3+x^5+x^9) \frac{d}{dx}\left(\frac{1}{1-x}\right) = (x^3+x^5+x^9)\frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$= (x^3+x^5+x^9)\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} nx^{n+2} + \sum_{n=0}^{\infty} nx^{n+4} + \sum_{n=0}^{\infty} nx^{n+8}.$$

Thus, for $n \geq 8$, a_n is equal to

$$(n-2) + (n-4) + (n-8) = 3n - 14.$$

Therefore, $a_{15} = 45 - 14 = 31$, so there are 31 solutions to the given equation with $n = 15$ and subject to the constraints.

(2) Since each integer may be used only once into a partition into distinct parts, $D(x) = \prod_{n=1}^{\infty}(1+x^n)$.
For odd parts, we have $O(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$.

Now since $(1+x^n)(1-x^n) = 1-x^{2n} \Rightarrow (1+x^n) = \frac{1-x^{2n}}{1-x^n}$,

we have

$$D(x) = \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = O(x),$$

so the number of partitions of an integer into distinct parts is equal to the number of partitions into odd parts.

(3) Since the given advancement operator polynomial is factored and the equation is homogeneous, we may simply read off the solution as

$$f(n) = c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 (-2)^n + c_5 + c_6 n + c_7 (-6)^n.$$

(4) We first notice that inside Q_n we see two copies of Q_{n-1}, the first consisting of vertices whose strings start with 0 and the second consisting of those whose strings start with 1. The only edges between these two copies are edges between vertices that differ only in the first digit. Since there are 2^{n-1} vertices in Q_{n-1}, there are 2^{n-1} such edges. Q_n then also has all the edges from each copy of Q_{n-1}, so

$$q_n = 2q_{n-1} + 2^{n-1} \Rightarrow q_{n+1} - 2q_n = 2^n$$

$$\Rightarrow (A-2)q = 2^n.$$

As a particular solution we try $f_0(n) = dn2^n$.

$$(A-2)(dn2^n) = d(n+1)2^{n+1} - 2dn2^n = d2^{n+1},$$

so $d = \frac{1}{2}$ and $f_0(n) = n2^{n-1}$. The general homogeneous solution is $q_1(n) = c2^n$, so $q(n) = c2^n + n2^{n-1}$. However $q(1) = 1 = c2^1 + 1 \cdot 2^0$, so $c = 0$ and $q(n) = n2^{n-1}$.

(5) We first endeavor to find a particular solution. Since 1 is not a root of the advancement operator polynomial, we need only $d_1 n + d_2$ for the $2n$. (Without the d_2, we get a system of equations that cannot be solved — it wants $d_1 = 0$ and $d_1 \neq 0$ simultaneously.) However, 2 is a root of the advancement operator polynomial, so we need $d_3 n 2^n$ for the $4 \cdot 2^n$. Thus we try

$$f_0(n) = d_1 n + d_2 + d_3 n 2^n$$

and have

$$(A-2)(A-3)[d_1 n + d_2 + d_3 n 2^n]$$

$$= (A-2)[d_1(n+1) + d_2 + d_3(n+1)2^{n+1} - 3d_1 n - 3d_2 - 3d_3 n 2^n]$$

$$= (A-2)[-2d_1 n + d_1 - 2d_2 - d_3 n 2^n + d_3 2^{n+1}]$$

$$= -2d_1(n+1) + d_1 - 2d_2 - d_3(n+1)2^{n+1} + d_3 2^{n+2} + 4d_1 n - 2d_1 + 4d_2 + 2d_3 n 2^n - 2d_3 2^{n+1}$$

$$= 2d_1 n - 3d_1 + 2d_2 - 2d_3 2^n$$

This gives $d_1 = 1$, so since $-3d_1 + 2d_2 = 0$, we have $-3 + 2d_2 = 0$ $\Rightarrow 2d_2 = 3 \Rightarrow d_2 = \frac{3}{2}$. Finally $d_3 = -2$. Thus

$$f_0(n) = n + \frac{3}{2} - 2n 2^n$$

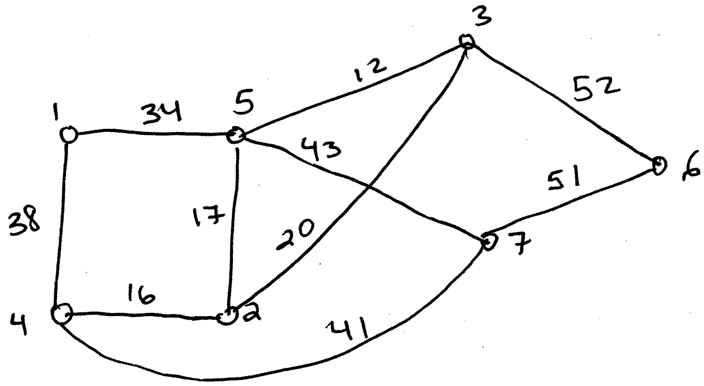
is a particular solution.

Solution for problem(s) 6 MATH 3012B Name: SOLUTION

(6) Mr. Duke realized the shipping can be modeled by a network. The source is Bentonville, Arkansas, the sink (appropriately described) is Athens, Georgia, (or at least their Wal-Mart). The roads are edges and the intermediate warehouses are vertices. The roads' weight limits are the edges' capacities. Mr. Duke then found an augmenting path to show Matthew S. that he hadn't thought hard enough (not surprising for a mutt). He also found a cut of smaller capacity than the one Knowshon M. was looking at around Athens, which showed his proposed 20 ton increase was impossible.

Use back for scratchwork. Back will not be graded. Use the front of an additional sheet if you need more space.

(7)



For Kruskal, we start by sorting the edges by weight:

53	12
42	16
25	17
23	20
15	34
14	38
47	41
75	43
67	51
36	52

Kruskal	Prim
53	15
42	53
25	52
15	42
47	47
67	76

8

$\sigma = (A, B, C, F, D, E)$

$d(A) = 0$ $P(A) = (A)$

B: $d(B) = 7$ $P(B) = (A, B)$

C: $d(C) = 17$ $P(C) = (A, C)$

$d(C) = 17$ $P(C) = (A, C)$

D: $d(D) = 55$ $P(D) = (A, D)$

$d(D) = 54$ $P(D) = (A, B, D)$

$d(D) = 43$ $P(D) = (A, C, D)$

$d(D) = 29$ $P(D) = (A, B, F, D)$

E: $d(E) = 83$ $P(E) = (A, E)$

$d(E) = 34$ $P(E) = (A, B, E)$

$d(E) = 34$ $P(E) = (A, B, E)$

$d(E) = 34$ $P(E) = (A, B, E)$

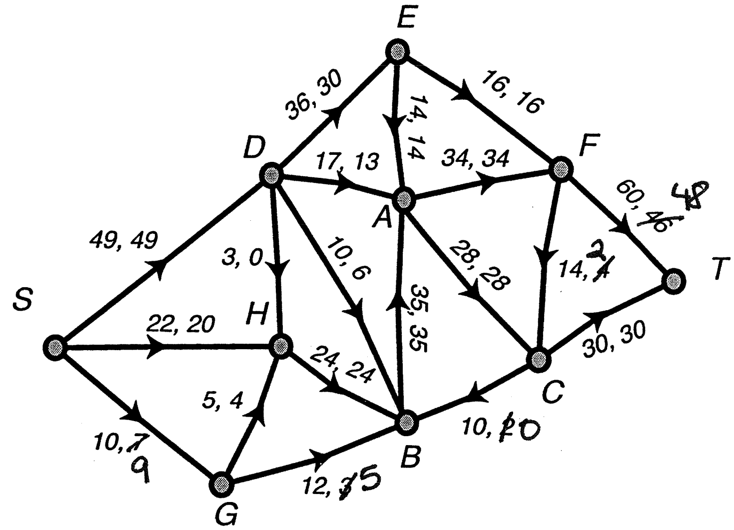
$d(E) = 33$ $P(E) = (A, B, F, D, E)$

F: $d(F) = 42$ $P(F) = (A, F)$

$d(F) = 24$ $P(F) = (A, B, F)$

$d(F) = 24$ $P(F) = (A, B, F)$

Use this figure and the space following for question 9:



We apply the Ford-Fulkerson Labeling Algorithm to find an augmenting path:

S: (*, +, ∞)
G: (S, +, 3)
H: (S, +, 2)
B: (G, +, 3)
C: (B, −, 2)
D: (B, −, 3)
A: (C, −, 2)
F: (C, −, 2)
E: (D, +, 3)
T: (F, +, 2)

P = (S, G, B, C, F, T)

$\delta = 2$

S: (*, +, ∞)
G: (S, +, 1)
H: (S, +, 2)
B: (G, +, 1)
D: (B, −, 1)
A: (D, +, 1)
E: (D, +, 1)

Thus, our minimum cut is

L = {S, G, H, B, D, A, E}

U = {F, C, T}, which has capacity

16 + 34 + 28 = 78.

New flow value = 78, and this flow is maximum since the algorithm didn't find an augmenting path on the next run.

Use back for scratchwork. Back will **not** be graded. Use the front of an additional sheet if you need more space.