MATH 3012G Final Exam

Spring 2010

Name: ______________________________

GTid (9xxxxxxxx): ______________________________

Group: ______________________________

Instructor: Mitchel T. Keller

There are 14 questions on this exam on 14 pages (not counting this coverpage). **Answer each question in the space provided. If you need additional space, additional pages will be provided. The back of pages will not be graded!** Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something or run an algorithm) will not receive full credit. Place your name and group on each page. Any page missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Student signature: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
Total:	70	

1. (5 points) A U[sic]GA football player is driving his scooter home from an establishment specializing in adult beverages one night. This establishment is at the intersection of 1st Street and 1st Avenue. He needs to return to his apartment at the intersection of 7th Street and 5th Avenue. However, one of his teammates has called him and informed him that Deputy Cletus Hogg is parked at the intersection of 4th Street and 4th Avenue. The football player wants to avoid elevating his team's arrest count, so he must avoid Deputy Hogg. Assuming he's sober enough to drive only on streets and avenues and follows as short of a path as possible (Yes, these are big assumptions for a U[sic]GA football player!), in how many ways can he get back to his apartment? (Streets and avenues are uniformly spaced and numbered consecutively in this hypothetical model of Athens.)

2. (5 points) Use the Euclidean algorithm to find the greatest common divisor d of 458 and 16. Find *integers* a and b so that $d = 458a + 16b$. Is it possible to find integers a' and b' so that $7 = 458a' + 16b'$? Why or why not?

3. (5 points) Recall from calculus the product rule: If f and g are differentiable functions, then

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g.$$

Also recall that $dx/dx = 1$. Use these facts and mathematical induction to prove that for every integer $n \geq 1$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

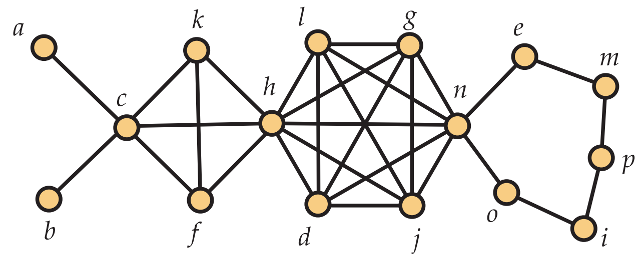
4. (5 points) Let m and w be positive integers. Give a combinatorial proof of the fact that for integers $k \geq 0$,

$$\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}.$$

Hint: Think of forming a committee.

5. (5 points) Construct the labeled tree with Prüfer code 4724214.

6. (5 points) Determine the number of spanning trees of the graph shown below.



7. (5 points) Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $\mathbf{P} = (X, P)$ be the poset so that $x \leq y$ in P if and only if x evenly divides y. (For example, $2 \leq 4$ in P, but 2 and 5 are incomparable in P.)

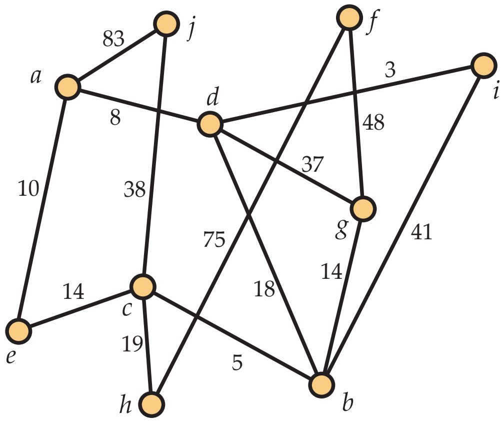
   (a) Draw the diagram of **P**.

   (b) List the maximal elements of **P**.

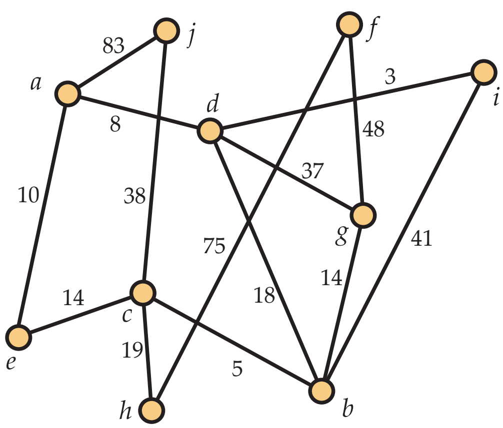
   (c) Find a maximal chain in **P** with two elements.

8. (5 points) Recall that $S(m,n)$ denotes the number of surjections (also known as onto functions) from $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,n\}$. Give the formula for $S(8,4)$ and explain why it is true. You may **not** simply appeal to the fact that the general formula was proved in the text/class.

9. (5 points) Use both Kruskal's algorithm (avoid cycles) and Prim's algorithm (build tree) to construct a minimum weight spanning tree in the graph below. (For Prim's algorithm, start with vertex a as the root.) For your answers, list the edges added by each algorithm in the order they are added along with the edge weights.



Kruskal (avoid cycles)



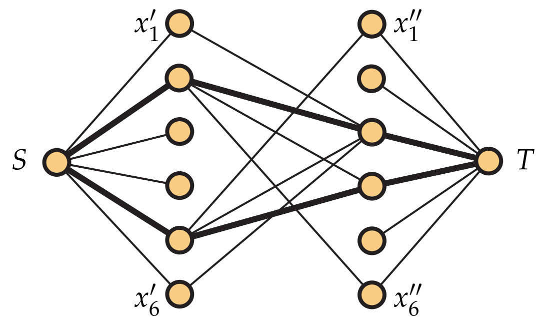
Prim (build tree)

10. (5 points) Find the general solution of the advancement operator equation

$$(A+1)^2(A-3)f = 6n.$$

11. (5 points) Let t_n be the number of ways to tile a $2 \times n$ rectangle using 1×1 tiles and L-tiles. An L-tile is a 2×2 tile with the upper-right 1×1 square deleted. (An L tile may be rotated so that the "missing" square appears in any of the four positions.) Find a recursive formula for t_n along with enough initial conditions to get the recursion started.

12. (5 points) Below is a network corresponding to a poset $\mathbf{P} = (X, P)$ with $X = \{x_1, \ldots, x_6\}$. In this network, all edges are directed from left to right and have capacity 1. The thin edges have flow 0, while the **bold** edges have flow 1. The vertices in the left column of six are x'_1 to x'_6 from top to bottom. The vertices in the second column of six are x''_1 to x''_6 from top to bottom. Use the Ford-Fulkerson labeling algorithm with pseudo-alphabetic order $S, T, x'_1, \ldots, x'_6, x''_1, \ldots, x''_6$ to determine the width w of $\mathbf{P}$, an antichain of size w, and a partition of $\mathbf{P}$ into w chains.



13. (5 points) If $\mathbf{G} = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, its *degree sequence* is the list $(\deg_{\mathbf{G}}(v_1), \deg_{\mathbf{G}}(v_2), \ldots, \deg_{\mathbf{G}}(v_n))$. Below are three sequences of length 10. One of the sequences cannot be the degree sequence of any graph. Identify it and say why. For each of the other two, say *why* (if you have enough information) a *connected* graph with that degree sequence

    (i) is definitely hamiltonian/cannot be hamiltonian;

    (ii) is definitely eulerian/cannot be eulerian;

    (iii) is definitely a tree/cannot be a tree; and

    (iv) is definitely planar/cannot be planar.

    (If you do not have enough information to make a determination for a sequence without having specific graph(s) with that degree sequence, write "not enough information" for that property.) The sequences are

    (a) $(6, 6, 4, 4, 4, 4, 2, 2, 2, 2)$

    (b) $(7, 7, 7, 7, 6, 6, 6, 2, 1, 1)$

    (c) $(8, 6, 4, 4, 4, 3, 2, 2, 1, 1)$

14. (5 points) Alice and Bob are attempting to determine the chromatic number of a graph $\mathbf{G} = (X, E)$. They are absolutely certain that the clique number of the graph is 7. Bob says that's enough to guarantee that the chromatic number is 7. Alice says that alone is not enough information. However, after some work, she finds a poset $\mathbf{P} = (X, P)$ so that x and y are comparable in P if and only if $xy \in E$. She then claims that by running an algorithm on the poset $\mathbf{P}$, she is able to find a proper coloring of the graph $\mathbf{G}$ using 7 colors. Explain how she does this. (You do not need to explain how she would come up with the poset, only how she would use the poset to color the graph.)