Group Name

MATH 3012G TEST II
TAKE-HOME PROBLEM

SPRING 2010

Instructions. You are to work on this problem completely alone. You are permitted to contact the instructor with questions, but otherwise are not to communicate with any other individual about this problem. **Attach your solution to this piece of paper with a staple. At the top of this sheet of paper, write your name and group.** You may use computer algebra systems, calculators, the course textbook and materials on T-Square, and your notes to solve this problem. Other than to access course materials on T-Square, to contact the instructor, or to use Wolfram|Alpha as a computer algebra system, you are not to use the Internet to assist in solving this problem. You must also fully explain all of the steps of your work using *complete sentences*. Your solution must rely on techniques discussed in this course to receive credit.

By signing on the line below, you certify that you have followed the rules above, including that you have not discussed this problem with anyone other than the instructor. You also certify that you have adhered to the Georgia Tech Honor Code, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to comply with the rules of this take-home problem will result in a report to the Office of Student Integrity.

Student signature: ______________________________

Due date. Your solution to this problem is due no later than when your group folder is submitted at the end of class on Friday, 2 April 2010, in class. Late work will not be accepted. In the event you cannot make it to class, send a digital copy to keller@math.gatech.edu before 1355 on 2 April 2010.

Problem. At a small school (such as the one your instructor attended), there is a class with nine students in it. The students, whom we will denote as A, B, C, D, E, F, G, H, and I, walk from their classroom to the lunchroom in the order $ABCDEFGHI$. (Let's say that A is at the front of the line.) On the way back to to their classroom after lunch, they would like to walk in an order so that no student walks immediately behind the same classmate he or she was behind on the way to lunch. (For instance, $ACBDIHGFE$ and $IHGFEDCBA$ would meet their criteria. However, they would not be happy with $CEFGBADHI$ since it contains FG and HI, so G is following F again and I is following H again.) One student ponders how many possible ways there would be for them to line up meeting this criterion. Help him out by determining the exact value of this number. (You can and should use Wolfram|Alpha (or a calculator or another CAS) to simplify the expression you develop into a single integer in the end.) Is this bigger than, smaller than, or equal to the number of ways they could return so that no student walks in the same position as before (i.e., A is not first, B is not second, ..., and I is not last)? What fraction (give it as a decimal, please) of the total number of ways they could line up meet their criterion of no student following immediately behind the same student on the return trip?

Some suggestions:

- Your goal is to count all ways to rearrange the students that do not contain XY, where Y followed immediately behind X on the way to lunch (i.e., XY is a substring of $ABCDEFGHI$ such as BC or GH). Think about how you might count those strings that *do* contain one or more pairs that are in the original order and use those number to achieve your goal. When counting rearrangements that contain more than one pair in the orginal order, be careful, as the same student might appear in two of them! (As an example, AB and BC might both appear in the "bad" rearrangement.) This will not be a major hurdle (as compared to if, for example, AB and DE both appear in a "bad" rearrangement), but you do need to discuss it.
- You're not going to find the answer by brute force. (Writing computer code to generate them all is out since you need to rely on techniques from this course.)

- You may find it helpful to start with a smaller case to see what's going on and try to find a pattern. The case of two students is not instructive, but the cases of three students and four students may be helpful (and you can quickly list all possibilities by hand).

Solution. One (but not the only) way to approach this problem is via inclusion-exclusion. To do this, we need to define a collection $\mathscr{P}$ of properties. These properties will be based having a pair of students in the original order. Thus, the properties are P_{AB} for having A followed immediately by B, P_{BC} for having B followed immediately by C, and so on up to P_{HI}. This gives 8 properties.

The total possible number of permutations of the students is clearly $9! = P(9,9)$. We now consider how many permutations satisfy property P_{AB}. To satisfy P_{AB}, a permutation must have AB as a substring. Thus, we may think of AB as a new "symbol". We remove A and B from our set of letters and thus have 8 symbols to permute, which can be done in 8! ways. This analysis is identical for strings satisfying any one of the other properties, so there are $C(8,1)8!$ strings satisfying one property.

For two properties, the situation is a bit different, as we must pay attention to whether the pairs being kept in their original order overlap. Suppose first that they don't. Then we might as well consider permutations satisfying both P_{AB} and P_{CD}. Here, we think of AB and CD as new symbols. We have eliminated four letters from our original 9 and added two new symbols, so we have $9-4+2=7$ things to permute. This can be done in 7! ways. The alternative is that the pairs overlap, so we consider permutations satisfying both P_{AB} and P_{BC}. In this case, we must have ABC appearing consecutively and treat it as a new symbol. Here, we have eliminated three letters and added one new symbol, giving $9-3+1=7$ things to permute. This can also be done in 7! ways. Therefore, the number of permutations satisfying two properties is independent of whether the pairs the properties represent overlap and there are $C(8,2)7!$ permutations satisfying two properties.

Now it is becoming clear what's going on. Let's consider the more general setting of having k properties satisfied. If those k properties do not overlap, then we eliminate $2k$ letters from our 9 and add k new symbols, leaving $9-2k+k = 9-k$ things to permute in $(9-k)!$ ways. What happens when pairs overlap? Suppose i letters appear in more than one pair. Then we eliminate $2k-i$ letters from our set of 9. The number of new symbols is $k-i$, since each of the i overlaps merges two symbols, eliminating a new one. That leaves us with $9-(2k-i)+k-i = 9-k$ things to permute, again in $(9-k)!$ ways. There are $C(8,k)$ ways to choose k properties from $\mathscr{P}$.

Now by the principle of inclusion-exclusion, we have that the number of ways for the students to walk back to class so that no one follows immediately behind the same student he/she followed to lunch is

$$\sum_{k=0}^{8}(-1)^k\binom{8}{k}(9-k)! = 148392.$$

This is 0.409 of the 9! total ways, so there's about a 41% chance that if they line up randomly they will be in an order that makes them happy. The number of ways they could walk back so that no student walks in the same position as before is the number of derangements of nine things $d_9 = 133496$, so there are more ways for them to walk back subject to their criterion than there would be ways for each to walk in a different position. $\square$