Group Name

MATH 3012G TEST III
TAKE-HOME PROBLEM

SPRING 2010

Instructions. You are to work on this problem completely alone. You are permitted to contact the instructor with questions, but otherwise are not to communicate with any other individual about this problem. **Attach your solution to this piece of paper with a staple. At the top of this sheet of paper, write your name and group.** You may use computer algebra systems, calculators, the course textbook and materials on T-Square, and your notes to solve this problem. Other than to access course materials on T-Square, to contact the instructor, or to use Wolfram|Alpha as a computer algebra system, you are not to use the Internet to assist in solving this problem. You must also fully explain all of the steps of your work using *complete sentences*. Your solution must rely on techniques discussed in this course to receive credit.

By signing on the line below, you certify that you have followed the rules above, including that you have not discussed this problem with anyone other than the instructor. You also certify that you have adhered to the Georgia Tech Honor Code, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to comply with the rules of this take-home problem will result in a report to the Office of Student Integrity.

Student signature: ____

Due date. Your solution to this problem is due no later than when your group folder is submitted at the end of class on Monday, 26 April 2010, in class. Late work will not be accepted. In the event you cannot make it to class, send a digital copy to keller@math.gatech.edu before 1355 on 26 April 2010.

Problem. Determine a generating function for the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = n$$

subject to the following conditions:

(1) $0 \leq x_1 \leq 3$
(2) $x_2 \geq 0$
(3) $x_3, x_4 \geq 0$ are multiples of 3 with $x_3 < 8$ (no upper bound on x_4)
(4) $x_5 \geq 5$ is a multiple of 5

Without doing any calculations (just by examining the conditions above), determine the coefficient on x^n for $n \leq 5$. Determine the coefficient on x^n for $n \geq 6$ by using a computer algebra system. Your solution will be best expressed by giving multiple formulas that depend on the remainders when n is divided by 3 and when n is divided by 5. (For instance, a formula for when n is divisible by both 3 and 5, another for when it is divisible by 3 and the remainder upon division by 5 is 4, etc.)

Suggestions: Wolfram|Alpha can be given the command `partial fractions` followed by your generating function to get a partial fractions expansion. However, there will be some issues with this expansion. Some of the terms W|A will readily give you general power series representations for. Others, it's less than helpful for. In those cases, I strongly suggest that you first group the terms that have the same denominator. Then you'll have a rational function $p(x)/q(x)$ (well, you might have a couple different denominators to deal with, so repeat as necessary). Focus on finding a series for $1/q(x)$, because then you can multiply that series by the polynomial $p(x)$. For instance, if you found $p(x) = 3x^4 - 3x^2 + 1$ and the series representation $\sum_{n=0}^{\infty} x^{7n}$ for $q(x)$, you could multiply and have

$$\sum_{n=0}^{\infty} 3x^{7n+4} + \sum_{n=0}^{\infty} (-3)x^{7n+2} + \sum_{n=0}^{\infty} x^{7n}.$$

(When you get something like this, notice that for the case when k is a multiple of 7 ($k = 7n$) the coefficient here on x^k is 1, when the remainder of dividing k by 7 is 2 ($k = 7n + 2$), the coefficient is -3, etc.)

The other thing you'll want to work with is rewriting your $p(x)/q(x)$ since W|A isn't good at finding series sometimes. If it gives you really ugly coefficients involving fractional powers of -1, there's a better way. In such an instance, remember that $(1 - x)(1 + x + x^2 + \cdots + x^k) = 1 - x^{k+1}$ and think about multiplying the numerator and denominator by the same thing before proceeding as above. In the process, you should find yourself working with a function $1/r(x)$ that you can readily recognize the series for. (In fact, it may still be one that W|A doesn't work well with, but it's a fundamental type of series we used all over in Chapter 7.)

As a final thought, remember that you can check your answer by using W|A's `series` command to get numerical values for coefficients as far out as you want. The "More terms" link will probably prove sufficient, but if not, see the slides for Chapter 7 for a more powerful command.