MATH 3012G Test I

Spring 2010

Name: ______________________________
GTid (9xxxxxxxx): ______________________________
Group: ______________________________
Instructor: Mitchel T. Keller

There are 6 questions on this exam on 4 pages (not counting this coverpage). **Answer each question on a separate solution sheet.** Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible**; answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name, group, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Student signature: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. (5 points) A database uses 20-character strings as record identifiers. The valid characters in these strings are upper-case letters in the English alphabet and decimal digits. (Recall there are 26 letters in the English alphabet and 10 decimal digits.) How many valid record identifiers are possible if a valid record identifier must meet **all** of the following criteria:

   - Letter(s) from the set $\{A, E, I, O, U\}$ occur in *exactly* three positions of the string.
   - The last three characters in the string are *distinct* decimal digits that do not appear elsewhere in the string.
   - The remaining characters of the string may be filled with any of the remaining letters or decimal digits.

   Solution: It's probably best to fill in the last three spaces with distinct decimal digits first, since that's the only rule that specifies what specific positions must be. There are $P(10,3) = 10 \cdot 9 \cdot 8$ ways to fill these positions. Now we shift our focus to the vowels. We need to choose which of the 17 remaining positions will house the three vowels, which can be done in $C(17,3)$ ways. Once chosen, there are 5^3 ways to fill those positions. Now there are 14 positions remaining in the string. We cannot use any vowels or the three digits we've already used, so that leaves 28 potential symbols. Thus, there are 28^{14} ways to fill the remaining positions. Therefore, there are

   $$P(10,3)\binom{17}{3}5^3 28^{14}$$

   valid record identifiers.

2. (5 points) A teacher has 450 identical pieces of candy. He wants to distribute them to his class of 65 students, although he is willing to take some leftover candy home. (He does not insist on taking any candy home, however.) The student who won a contest in the last class is to receive at least 10 pieces of candy as a reward. Of the remaining students, 34 of them insist on receiving at least one piece of candy, while the remaining 30 students are willing to receive no candy.

   (a) In how many ways can he distribute the candy?

   Solution: We begin by noting that the teacher will also get a pile, so we will be dividing the candy up into 66 piles, meaning we will eventually choose 65 gaps into which dividers are placed. We set aside 9 pieces of candy for the overachiever. We also add 30 artificial pieces of candy for the students who are willing to not get any candy and another for the teacher. Thus, we consider distributing $450 - 9 + 30 + 1 = 472$ pieces of candy. These determine 471 gaps, so there are

   $$\binom{471}{65}$$

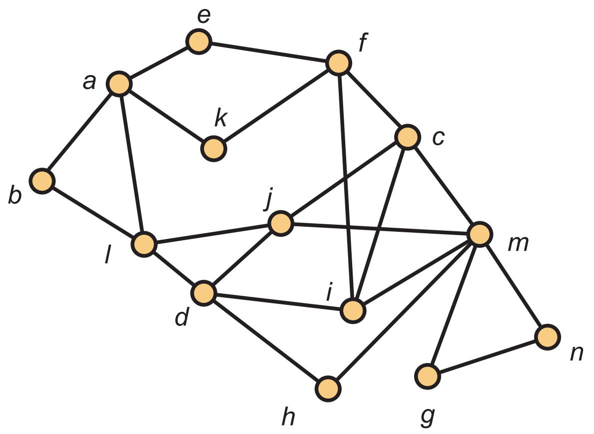
   ways to distribute the candy.

   (b) In how many ways can he distribute the candy if, in addition to the conditions above, one of his students is diabetic and can receive at most 7 pieces of candy? (This student is one of the 34 who insist on receiving at least one piece of candy.)

Solution: Here we just need to eliminate the ways of distributing the candy from the previous part where the diabetic student gets **more than** seven pieces of candy. To do this, we set aside seven pieces of candy for the student. This leaves 465 pieces of candy to distribute, so there are $C(464,65)$ "bad" distributions (or as one of you wrote "ways to kill the diabetic kid"). Subtracting these from the previous part gives the acceptable distributions, so we have

$$\binom{471}{65} - \binom{464}{65}.$$

3. (5 points) Let **G** be the graph below. (A special solution sheet containing **G** is provided.)



(a) Determine if **G** is eulerian. If it is, find an eulerian circuit. If it is not, give a reason why it is not.

Solution: Yes, the graph is eulerian, since the degree of every vertex is even. The circuit found by our algorithm is

$$a,e,f,c,i,d,h,m,c,j,d,l,j,m,g,n,m,i,f,k,a,b,l,a.$$

(First you find a,b,l,a and then expand at a with the rest.)

(b) Determine if **G** is hamiltonian. If it is, find a hamiltonian cycle. If it is not, give a reason why it is not.

Solution: No, the graph is not hamiltonian. There are a couple of good reasons for this. One reason is the triangle induced by $\{g,m,n\}$. If there were a hamiltonian cycle, we could make it start at g. We'd have to leave the triangle via m, but then there'd be no way to get back to g to close off the cycle. The other thing to look at is $\{a,e,f,k\}$. The edges ae, ef, fk, and ak must be on any hamiltonian cycle (only way in and out of e and k), but this is a $\mathbf{C}_4$, so we couldn't build a bigger cycle containing all those edges.

Note that Dirac's theorem does not apply here! Dirac's theorem simply says that **if** every vertex has at least $\lceil n/2 \rceil$ neighbors (in an n-vertex graph), **then** it is hamiltonian. The graph $\mathbf{C}_n$ is hamiltonian for every n even though every vertex has degree 2.

4. (5 points) Use mathematical induction to prove that for every integer $n \geq 0$,

$$n^3 + (n+1)^3 + (n+2)^3$$

is a multiple of 9. (Recall that an integer m is a multiple of 9 if $m = 9s$ for some integer s.)

Solution: We begin with the base case $n = 0$. For this, we see that $0^3 + 1^3 + 2^3 = 9$, which is a multiple of n. Now suppose that for some $k \geq 0$ that $k^3 + (k+1)^3 + (k+2)^3 = 9s$ is a multiple of 9. Consider

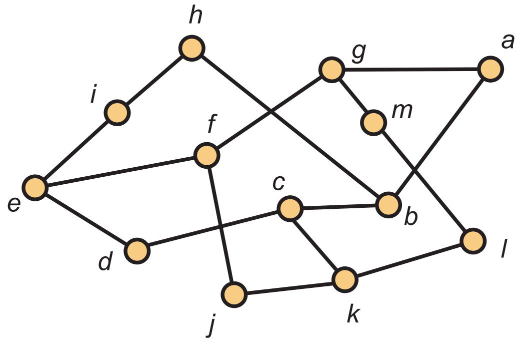
$$(k+1)^3 + (k+2)^3 + (k+3)^3 = (k+1)^3 + (k+2)^3 + k^3 + 9k^2 + 27k + 27.$$

By the induction hypothesis, the first three terms sum to $9s$. Thus, we have

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9s + 9k^2 + 27k + 27 = 9(s + k^2 + 3k + 3) = 9s',$$

letting s' be the integer $s + k^2 + 3k + 3$. Thus $(k+1)^3 + (k+2)^3 + (k+3)^3$ is a multiple of 9, and the fact is proved by mathematical induction.

5. (5 points) Find the chromatic number $\chi(\mathbf{G})$ of the graph $\mathbf{G}$ below as well as a proper coloring of $\mathbf{G}$ using $\chi(\mathbf{G})$ colors. (A special solution sheet containing $\mathbf{G}$ is provided.)



Solution: There are lots of odd cycles in this graph, so the graph is not bipartite. In other words, $\chi(\mathbf{G}) \geq 3$. One specific odd cycle is a, b, c, d, e, f, g. There is a proper 3-coloring of the graph, so $\chi(\mathbf{G}) = 3$. In the interest of posting this sooner rather than later, I won't mark up such a coloring unless requested.

6. (5 points) Give a combinatorial proof of the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

No credit will be awarded for proofs relying on algebraic manipulations!
(*Hint*: Think about choosing a team with a captain.)

Solution: Think of the number of ways to choose a team of k players from a class of n students with the requirement that one member of the team be designated as the captain. On the left-hand side, we count the number of ways to do this by first selecting the k players from the team in $C(n,k)$ ways. Once this has been done, there are k ways to choose one of the k players to be the captain.

For the right-hand side, suppose that we first pick the captain from the entire class. There are n students, so there are n ways to choose the captain. Once the captain is chosen, there are $n-1$ students left to choose from, and we need to choose $k-1$ of them to join the captain and form a team of size k. There are $C(n-1,k-1)$ ways to do this.

Since the left-hand side and right-hand side count the same thing in different ways, they are equal.