

MATH 3012G Test II

Spring 2010

Name: _____

GTid (9xxxxxxx): _____

Group: _____

Instructor: Mitchel T. Keller

There are 6 questions on this exam on 4 pages (not counting this coverage). **Answer each question on a separate solution sheet.** Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. **Use complete sentences wherever possible;** answers that do not contain at least one complete sentence of explanation (and do not just ask for a list or for you to label something) will not receive full credit. Place your name, group, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

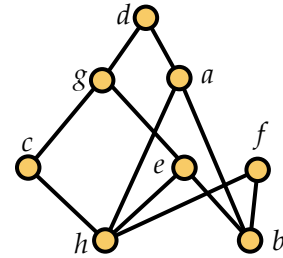
By signing on the line below, you agree to abide by the Georgia Tech Honor Code and Student Code of Conduct, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Student signature: _____

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. (5 points) Determine if the poset shown at the right is an interval order. If it is, find an interval representation using the algorithm from the book/class. If it is not, say why it is not. (Use the special solution sheet for this question!)



Solution: We begin by writing down the down-sets and up-sets:

$D(a) = \{h, b\}$	$U(a) = \{d\}$
$D(b) = \{ \}$	$U(b) = \{f, a, e, g, d\}$
$D(c) = \{h\}$	$U(c) = \{g, d\}$
$D(d) = \{a, g, c, e, h, b\}$	$U(d) = \{ \}$
$D(e) = \{h, b\}$	$U(e) = \{g, d\}$
$D(f) = \{h, b\}$	$U(f) = \{ \}$
$D(g) = \{c, e, h, b\}$	$U(g) = \{d\}$
$D(h) = \{ \}$	$U(h) = \{c, e, f, g, a, d\}$

Now we put them in order, removing duplicates:

$$\{ \} \subseteq \{h\} \subseteq \{h, b\} \subseteq \{c, e, h, b\} \subseteq \{a, g, c, e, h, b\}$$

$$D_1 \subseteq D_2 \subseteq D_3 \subseteq D_4 \subseteq D_5$$

$$\{c, e, f, g, a, d\} \supseteq \{f, a, e, g, d\} \supseteq \{g, d\} \supseteq \{d\} \supseteq \{ \}$$

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq U_4 \supseteq U_5$$

We are now prepared to identify the intervals. For a , we find that $D(a) = D_3$ and $U(a) = U_4$, so the interval $I(a) = [3, 4]$. Repeating this process for all eight points, we find the following:

$I(a) = [3, 4]$	$I(e) = [3, 3]$
$I(b) = [1, 2]$	$I(f) = [3, 5]$
$I(c) = [2, 3]$	$I(g) = [4, 4]$
$I(d) = [5, 5]$	$I(h) = [1, 1]$

2. (5 points) Lew the stand-up comic has developed a set of eight new math jokes. He performs seven nights per week at a local comedy club. (Yes, there's a big demand for comedian-mathematicians where he lives!) Lew wants to tell a subset of these eight jokes each night. He also wants to ensure that if someone is in the audience on *any* two different nights, he/she will hear at least one joke on each of the nights that was not performed on the other night. Lew has determined that it will take him 11 weeks to develop any new jokes to perform.

Does he have enough jokes to put on the type of shows he wants to? If so, why? If not, what's the largest number of shows he can perform that meet his criteria?

Solution: All of the "set" language in this question has me thinking about the subset lattice. Lew's eight jokes can be thought of as a set eight things. Thus, all the possible subsets he can tell on a night are subsets of this set, so we can model his shows as elements of 2^8 . To meet his criterion of allowing audience member who attend twice to always hear at least one joke each night that was not heard on the other night, the subsets of jokes he uses for his shows must form an antichain in 2^8 . We know by Sperner's Theorem that the largest antichain in 2^8 is formed by taking all subsets of size $8/2 = 4$. This antichain has size

$$\binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 2 \cdot 7 \cdot 5 = 70.$$

Thus, Lew will run out of material before his new jokes are written, since he can only perform the types of shows he wants to for 10 weeks.

3. (5 points) Dave and Noah have been given a large graph and told to determine if it is planar. Dave starts looking for a subdivision of K_5 or $K_{3,3}$. He's stressed because the graph has so many vertices that it will take him at least a week to check. Noah quickly notes that every vertex has degree at least six and tells Dave to stop, since he knows that the graph is not planar. Why is Noah right? (*Hint:* Think about the number of edges.)

Solution: Let n be the number of vertices in the graph. Noah notes that if every vertex has degree at least six, then the sum of the degrees of the vertices is at least $6n$. Since the sum of the degrees is twice the number of edges, the graph has at least $3n$ edges. We know that if a graph on n vertices is planar, it has at most $3n - 6$ edges. Therefore, there is no way this graph is planar. (Note: This allows us to conclude that any planar graph must have a vertex of degree at most 5.)

4. (5 points) For this question, the poset \mathbf{P} is shown at the right. (Use the special solution sheet for this question!)
 (a) List the minimal elements of \mathbf{P} .

Solution: The minimal elements of \mathbf{P} are $\{e, x, q, g\}$.

- (b) Find a maximal antichain containing three points. Is this antichain maximum? Why or why not?

Solution: The set $\{d, c, b\}$ is an antichain containing three points. It is maximal because it consists of all of the maximal elements of \mathbf{P} , so any other point is less than one of these points. It is not maximum since there are larger antichains, for instance $\{e, x, q, g\}$.

- (c) Use the algorithm from the book/class to find the height h of \mathbf{P} , a chain in \mathbf{P} containing h points, and a partition of \mathbf{P} into h antichains. (You should give your antichains by labeling the points in antichain A_1 by 1, those in A_2 by 2, etc. Do this on the copy of \mathbf{P} on the special solution sheet provided.)

Solution: The height of this poset is 8, as evidenced by the chain $\{e, f, w, v, p, l, j, d\}$. The antichains obtained by the algorithm are as follows:

$$A_1 = \{e, x, q, g\}$$

$$A_2 = \{f, k, u\}$$

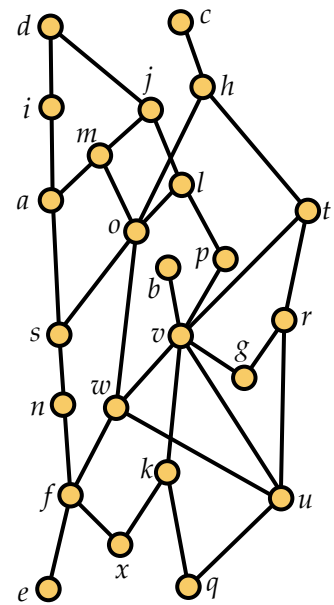
$$A_3 = \{n, w, r\}$$

$$A_4 = \{s, v\}$$

$$A_5 = \{a, o, b, p, t\}$$

$$A_6 = \{m, i, l, h\}$$

$$A_7 = \{j, c\}$$

$$A_8 = \{d\}$$


5. (5 points) Note that $442779777 = 3^5 \cdot 11^3 \cdot 37^2$. Determine $\phi(442779777)$. (You do not need to simplify your answer.)

Solution: By the formula developed in class, we know that

$$\phi(442779777) = 3^5 \cdot 11^3 \cdot 37^2 \cdot \frac{2}{3} \cdot \frac{10}{11} \cdot \frac{36}{37} = 261098640.$$

(The simplified answer is provided for amusement only. It is not required for full credit.)

6. (5 points) Alice and Bob are considering a poset with 39 points. Bob reports that after working really hard, he's found an antichain with 5 points in it. He then claims this tells them that the width of the poset is 5. Alice agrees that his set is an antichain. However, she tells him that she's unconvinced the width is 5. Should Alice be convinced? If so, why? If not, what must Bob produce to convince her that he's right?

Solution: No, Alice should not be convinced. It's possible that Bob missed an antichain with 6 or more points in it. To convince Alice that the width is 5, he must produce a partition of the 39 points of the poset into 5 chains. Dilworth's Theorem guarantees that the width is equal to the smallest number of chains into which the poset can be partitioned.