MATH 1501G1/G2/G3 Test I

Fall 2006

Name and GTid (9xxxxxxxx): ____________________

Instructor: Mitchel T. Keller

Teaching Assistant and Section: ____________________

There are 5 questions on this exam on 4 pages (not counting this coverpage). Answer each question on a separate solution sheet (you may use more than one solution sheet per problem if needed). Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. Place your name, section, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

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I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

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Student signature: ____________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
Total:	25	

1. (5 points) Using the ε, δ method, show that

$$\lim_{x \to -1} (-4x + 2) = 6.$$

Solution: Let $\varepsilon > 0$. We seek a $\delta > 0$ so that if $0 < |x - (-1)| < \delta$, then $|(-4x + 2) - 6| < \varepsilon$. To do this, note that

$$|(-4x + 2) - 6| = |-4x - 4| = |-4(x + 1)| = |-4||x + 1| = 4|x + 1|.$$

Since we want this to be less than ε when $0 < |x + 1| < \delta$, this suggests that we try $\delta = \varepsilon/4$.

We now show that this δ "works." To do this, we see that if $0 < |x + 1| < \delta = \varepsilon/4$, then

$$\begin{aligned} \varepsilon &> 4|x + 1| \\ &= |-4||x + 1| \\ &= |-4x - 4| \\ &= |-4x + 2 - 6|, \end{aligned}$$

which is what we desired. Therefore, the proof is complete.

2. (5 points) Consider the sequence a_n defined by

$$a_n = \begin{cases} 2^{-(n+1)/2} & n \text{ odd;} \\ 3^{-n/2} & n \text{ even.} \end{cases}$$

Is the sequence monotonic? If so, what type of monotonicity does it exhibit? If not, why is it not monotonic? Does the sequence a_n converge? If so, state its limit and explain why that is the limit. If not, explain why it diverges.

Solution: We write out the first few terms of the sequence and see that they are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}, \frac{1}{8}, \ldots.$$

Since $a_3 = 1/4 > 1/9 = a_4$ and $a_4 = 1/9 < 1/8 = a_5$, we see that the sequence is *not* monotonic.

The sequence does, however, converge to 0. To see this, note that for $n = 1, 2, \ldots$ we have $0 < a_n < 1/n$. We know that $1/n \to 0$ as $n \to \infty$, and thus by the pinching theorem for sequences, we have that $a_n \to 0$ as $n \to \infty$.

3. (5 points) Determine if the limit

$$\lim_{t\to 0} \frac{t^2\sin(3t)+6t^4-t^3\cos(2t)}{\tan(3t)(1-\cos(at))}$$

exists. If it does, find its value, and if it does not, explain why. (Assume that $a \neq 0$.)

Solution: This is an indeterminate form of type 0/0. Thus, we can't just evaluate the target function at 0 to compute the limit. To compute the limit, we thus begin by converting the $1-\cos(at)$ to a $\sin^2(at)$ by multiplying the numerator and denominator by $1+\cos(at)$, since $(1-\cos(at))(1+\cos(at)) = 1-\cos^2(at) = \sin^2(at)$. This converts our limit to

$$\lim_{t\to 0}\left[\frac{1+\cos(at)}{\sin^2(at)}\frac{t^2\sin(3t)+6t^4-t^3\cos(2t)}{\tan(3t)}\right].$$

We proceed to factor a t^2 out of the numerator and multiply numerator and denominator by a^2 (so that we have a pair of at's in the numerator to so with the $\sin(at)$'s in the denominator) and have that our limit is now

$$\lim_{t\to 0}\left[\frac{1+\cos(at)}{a^2}\frac{a^2t^2}{\sin^2(at)}\frac{\sin(3t)+6t^2-t\cos(2t)}{\tan(3t)}\right].$$

We now know the limit of the first factor is $2/a^2$ and the limit of the second factor is $1\cdot 1$, so if we can show that the third factor has a limit as $t\to 0$, we will have the total limit. To do this, we write the last fraction as a sum of three terms and have that our limit is now

$$\lim_{t\to 0}\left[\frac{1+\cos(at)}{a^2}\frac{a^2t^2}{\sin^2(at)}\left(\frac{\sin(3t)\cos(3t)}{\sin(3t)}+\frac{6t^2\cos(3t)}{\sin(3t)}-\frac{t\cos(2t)\cos(3t)}{\sin(3t)}\right)\right].$$

In the first term of the sum, the $\sin(3t)$'s cancel, making that term just $\cos(3t)$, which goes to 1 as $t\to 0$. In the middle term, we see that we have a factor of $t/\sin(3t)$, which if we multiply numerator and denominator by 3 will give us a factor that tends to 1 as $t\to 0$. We also note that $\lim_{t\to 0}\cos(3t) = 1$. However, in that term we still have a factor of t, which goes to 0 as $t\to 0$, so the middle term tends to 0. (Symbolically, we rewrite the middle term as

$$\frac{6t}{3}\frac{3t}{\sin(3t)}\cos(3t)$$

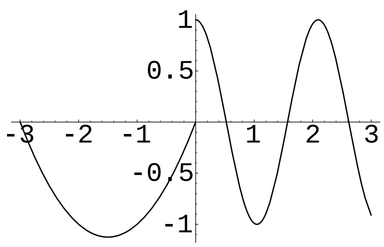
and see that its limit is 0.) Finally, in the last term, we need a factor of 3 in the numerator (and thus introduce one in the denominator as well) in order to have $3t/\sin(3t)$. The cosine factors both have limit 1 as $t\to 0$, and thus the limit of the last term is $1/3$. (Symbolically, we write the third term as

$$\frac{1}{3}\frac{3t}{\sin(3t)}\cos(2t)\cos(3t),$$

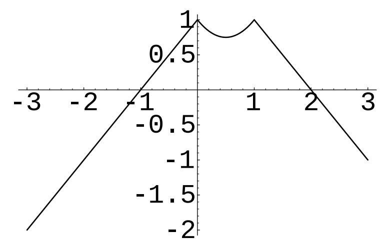
and observe that its limit is 1/3.) Hence, the limit does in fact exist and is

$$\frac{2}{a^2}\left(1-\frac{1}{3}\right)=\frac{4}{3a^2}.$$

4. (5 points) Consider the functions pictured below. (Take $f(0) = 1$.)



$f(x)$



$g(x)$

For what values of x is the function $f(g(x))$ continuous? For what values of x is the function $g(f(x))$ continuous? Justify your answers.

Solution: We first consider $f(g(x))$. We know from the theorem on the continuity of composition of functions that $f(g(x))$ is continuous at c if g is continuous at c and f is continuous at $g(c)$. Since f is continuous on all of $[-3, 3]$ except at 0, we automatically have that $f(g(x))$ is continuous anywhere that $g(x)$ is not 0, and thus we have that $f \circ g$ is continuous *at least* on $[-3, -1) \cup (-1, 2) \cup (2, 3]$. It remains to investigate the continuity of $f \circ g$ at -1 and 2. As x approaches -1 from the left, $g(x)$ approaches 0 from the left (below), and thus $f(g(x))$ approaches 0 as x approaches -1 from the left. As x approaches -1 from the right, $g(x)$ approaches 0 from above, and thus $f(g(x))$ approaches 1 as x approaches -1 from the right. Hence, $f \circ g$ is discontinuous at -1. Similarly,

$$\lim_{x\to 2^+} f(g(x)) = 0 \qquad \text{and} \qquad \lim_{x\to 2^-} f(g(x)) = 1,$$

and thus $f(g(x))$ is discontinuous at 2.

We now look at $g \circ f$. Since g is continuous on the entire interval, we can focus on how the continuity of f impacts us. When f is continuous at c, we get that $g \circ f$ is continuous at c, so thus we automatically have that $g \circ f$ is continuous on $[-3, 0) \cup (0, 3]$. Now as $x \to 0^-$, we have that $f(x)$ approaches 0 from the left and thus $g(f(x))$ approaches 1. As $x \to 0^+$, we have $f(x)$ approaches 1 from below, and thus $g(f(x))$ approaches 1. That is, we have

$$\lim_{x\to 0^-} g(f(x)) = 1 = \lim_{x\to 0^+} g(f(x)),$$

and further $g(f(0)) = g(1) = 1$, so therefore $g \circ f$ is continuous at 0, and thus is continuous on all of $[-3, 3]$.

5. (5 points) Show that there is a number $c \in [1, 4]$ such that

$$\csc\left(\frac{\pi}{3}c\right) = \frac{c^2}{\sqrt{3}}.$$

Solution: Let

$$f(x) = \csc\left(\frac{\pi}{3}x\right) - \frac{x^2}{\sqrt{3}}.$$

We need to show that there is a root of f in the interval $[1, 4]$. We first note that f is not continuous at 3, as $\csc(\pi)$ is undefined. Thus, we need to find a smaller interval if we hope to use the intermediate value theorem. Notice that $\csc((\pi/3)x)$ is defined and continuous on $[1, 2]$, as the sine is not zero on $[\pi/3, 2\pi/3]$. Now

$$f(1) = \csc(\pi/3) - 1/\sqrt{3} = 2/\sqrt{3} - 1/\sqrt{3} = 1/\sqrt{3}$$

and

$$f(2) = \csc(2\pi/3) - 4/\sqrt{3} = 2/\sqrt{3} - 4/\sqrt{3} = -2/\sqrt{3}.$$

Hence $f(1) > 0 > f(2)$, so by the intermediate value theorem, we know that there is $c \in [1, 2]$ (and thus in $[1, 4]$) such that

$$\csc\left(\frac{\pi}{3}c\right) = \frac{c^2}{\sqrt{3}}.$$