MATH 1501G1/G2/G3 Test II

Fall 2006

Name: ____

GTid (9xxxxxxxx): ______________________________________

Instructor: Mitchel T. Keller

Teaching Assistant and Section: __________________________

There are 6 questions on this exam on 4 pages (not counting this coverpage). Answer each question on a separate solution sheet (you may use more than one solution sheet per problem if needed). Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. Place your name, section, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, you will receive a score of zero (0) points on this exam.**

Student signature: ______________________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. (5 points) Using the definition of the derivative, find the derivative of $f(x) = \dfrac{1}{x^2}$, $x \neq 0$.

Solution: We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h} = \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} \\ &= \lim_{h\to 0} \frac{-2xh+h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \frac{-2x+h}{(x+h)^2x^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}. \end{aligned}$$

2. (5 points) Compute $f'(x)$ if

$$f(x) = \sin(2x) + \sqrt{x}(x^2-4)^{9/8} + \cos\left(\frac{\tan(4x^2-3)}{x^3+8}\right).$$

Solution: The first term requires only the chain rule, and so we have $(\sin(2x))' = \cos(2x)\cdot 2$. We need the product rule, power rule, and chain rule to do the middle term. This gives

$$\frac{d}{dx}\left(\sqrt{x}(x^2-4)^{9/8}\right) = \sqrt{x}\cdot\frac{9}{8}(x^2-4)^{1/8}\cdot 2x + \frac{1}{2\sqrt{x}}(x^2-4)^{9/8}.$$

The final term is the most problematic. For ease of writing up, let

$$u = \frac{\tan(4x^2-3)}{x^3+8}.$$

Then the derivative of the last term is $-\sin(u)\frac{du}{dx}$ where

$$\frac{du}{dx} = \frac{(x^3+8)\sec^2(4x^2-3)\cdot 8x - \tan(4x^2-3)(3x^2)}{(x^3+8)^2}.$$

Thus, the derivative is

$$f'(x) = \cos(2x)\cdot 2 + \sqrt{x}\cdot\frac{9}{8}(x^2-4)^{1/8}\cdot 2x + \frac{1}{2\sqrt{x}}(x^2-4)^{9/8} - \sin(u)\frac{du}{dx},$$

where u and $\frac{du}{dx}$ are as given above.

3. (5 points) Boyle's law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant ($PV = C$). A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain your answer.

Solution: We're given the relationship between pressure and volume, namely $PV = C$. We're interested in the rate at which the volume is changing, which is $\frac{dV}{dt}$. We're given that $\frac{dP}{dt}$ is constant, so that should help us out. We first rewrite our given relationship as $V = C/P$. Then

$$\frac{dV}{dt} = -\frac{C}{P^2}\frac{dP}{dt}.$$

But now $P = C/V$ from the given relationship, so we have

$$\frac{dV}{dt} = -\frac{C}{(C/V)^2}\frac{dP}{dt} = -\frac{V^2}{C}\frac{dP}{dt}.$$

Note that $\frac{dP}{dt}$ is constant, so the rate of change of volume depends only on the volume itself now. Since we are compressing the gas, the volume is smaller at the end of the 10 minutes, so $\frac{dV}{dt}$ is closer to 0 at the end of the 10 minutes. Therefore, the volume decreases more rapidly at the beginning of the 10 minutes.

4. (5 points) A rectangular poster is to have an area of 250 square inches with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

Solution: Suppose that the poster has width x and height y. Then the width of the printed area is $x - 2$ (subtract off 1 inch on both the left and the right) and the height of the printed area is $y - 3$ (subtract off 2 inches on the top and 1 inch on the bottom). We know that the area of the poster is $xy = 250$. The printed area is $A = (x - 2)(y - 3)$. But we know that $y = 250/x$, so we now have

$$A(x) = (x-2)\left(\frac{250}{x} - 3\right) = 250 - 3x - \frac{500}{x} + 6 = 256 - 3x - \frac{500}{x}.$$

The domain of A is $(0, \infty)$, so to find the maximum value, we find

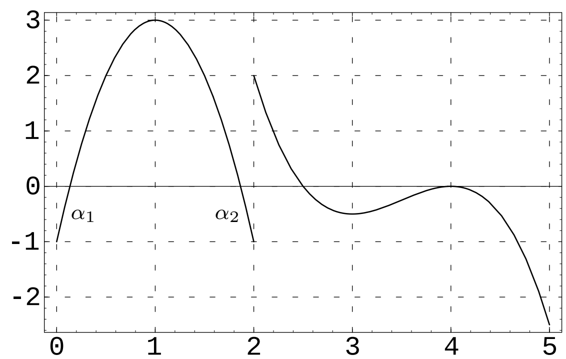
$$A'(x) = -3 + \frac{500}{x^2}.$$

We're interested in the critical numbers. Our first possibility is 0 since $A'(0)$ is undefined, but 0 is not in the domain of our function. Thus, we solve

$$0 = -3 + \frac{500}{x^2} \Rightarrow x^2 = \frac{500}{3} \Rightarrow x = \sqrt{\frac{500}{3}} = 10\sqrt{\frac{5}{3}}.$$

(The negative root is not in the domain of f, so this is the only point we have to care about.) To the left of this value, the derivative is positive, and to the right, it is negative, so we do in fact have a local maximum at $x = 10\sqrt{5/3}$. We now just have to compute $y = 250/x = 250/(10\sqrt{5/3}) = 25\sqrt{3/5}$. Therefore, the dimensions of the poster that will maximize the printed area are width of $10\sqrt{5/3}$ and height of $25\sqrt{3/5}$.

5. (5 points) The figure below shows **the first derivative** of a continuous function f with domain $\{x\colon 0 \leq x \leq 5\}$. State (1) the intervals where f increases and decreases, (2) where f has local maxima and local minima, (3) where f is concave up and concave down, and (4) where f has points of inflection.



Solution: We know that a function increases when its derivative is positive and decreases when it is negative. Thus, f increases on $[\alpha_1, \alpha_2]$ and $[2, 5/2]$ and f decreases on $[0, \alpha_1]$, $[\alpha_2, 2]$, and $[5/2, 5]$. (Recall from the text that a function decreases on an interval if its derivative is strictly negative on that interval except for being zero at finitely many points.)

Local maxima and local minima occur where the derivative changes sign. Since f' goes from negative to positive at α_1, there is a local minimum at α_1. At α_2, the derivative goes from positive to negative, so we have a local maximum there. At $5/2$, the same thing occurs, so there is another local maximum at $5/2$. Since we are given that f is continuous, 2 is also a critical number (the derivative does not exist there) and to the left f' is negative while it is positive to the right. Thus, there is a local minimum at 2 as well.

The graph of a function f is concave up when its derivative f' increases, so the graph of f is concave up on $(0, 1)$ and $(3, 4)$. On the other hand, the graph is concave down when its derivative f' decreases, so the graph of f is concave down on $(1, 2)$, $(2, 3)$, and $(4, 5)$.

A function has a point of inflection if its concavity changes there. Thus, the points of inflection for f are $(1, f(1))$, $(3, f(3))$, and $(4, f(4))$.

6. (5 points) Alice and Bob are studying for their calculus test. Bob claims that he knows of a function $f(x)$ that is differentiable for all x and satisfies $f'(x) < 2$ for all $x \in (1, 3)$. Furthermore, he says that at the points below, f takes on the following values:

x	$f(x)$
1	3
2	4
3	6

(The table does not define f. It only specifies values at a few points.) Alice says Bob doesn't know what he's talking about. Who is correct? Carefully explain why that individual is correct.

Solution: Alice is correct. Notice that

$$\frac{f(3) - f(2)}{3 - 2} = \frac{6 - 4}{1} = 2.$$

If Bob were correct, his function f would be continuous on $[2, 3]$, since it is differentiable for all x, which implies that it is continuous for all x. Thus, f would be continuous on $[2, 3]$ and differentiable on $(2, 3)$, so we would have by the Mean Value Theorem that there is a $c \in (2, 3) \subseteq (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(2)}{3 - 2} = 2,$$

contrary to Bob's claim that $f'(x) < 2$ for all $x \in (1, 3)$.