MATH 1501G1/G2/G3 Test III

Fall 2006

Name: ______________________________

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There are 6 questions on this exam on 4 pages (not counting this coverpage). Answer each question on a separate solution sheet (you may use more than one solution sheet per problem if needed). Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. Place your name, section, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

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I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, three points will be deducted from your raw total score on this exam.**

Student signature: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. (5 points) Find $F'(x)$ if

$$F(x) = \int_{8\arctan(\sqrt{x}+4\pi)}^{e} t^4 \ln t \, dt.$$

Solution: We use the chain rule combined with the first part of the Fundamental Theorem of Calculus to compute this derivative after multiplying by -1 in order to put the limit of integration that contains our variable as the upper limit. When we do so, we have

$$F'(x) = -(8\arctan(\sqrt{x}+4\pi))^4 \ln(8\arctan(\sqrt{x}+4\pi)) \cdot 8 \cdot \frac{1}{1+(\sqrt{x}+4\pi)^2} \cdot \frac{1}{2\sqrt{x}}.$$

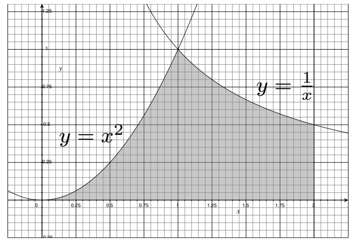
2. (5 points) Find dy/dx if $x^y = y^x$.

Solution: We use logarithmic differentiation and implicit differentiation to solve this problem. We first take the logarithm of both sides, using absolute values to ensure that the logarithm is defined, and then differentiate. We have

$$\begin{aligned} & x^y = y^x \\ \Rightarrow \quad & y\ln|x| = x\ln|y| \\ \Rightarrow \quad & y\frac{1}{x} + \frac{dy}{dx}\ln|x| = x\frac{1}{y}\frac{dy}{dx} + \ln|y| \\ \Rightarrow \quad & \frac{dy}{dx}\left(\ln|x| - \frac{x}{y}\right) = \ln|y| - \frac{y}{x} \\ \Rightarrow \quad & \frac{dy}{dx} = \frac{\ln|y| - \frac{y}{x}}{\ln|x| - \frac{x}{y}}. \end{aligned}$$

3. (5 points) Find the area of the region bounded by the curves $y = x^2$, $y = 1/x$, $y = 0$, and $x = 2$.

Solution: The region is shown in the figure below. From this, we see that it is bounded above by $y = x^2$ from 0 to 1 and then by $y = 1/x$ from 1 to 2.



Thus, the area of the region is

$$\int_0^1 x^2\,dx + \int_1^2 \frac{dx}{x} = \left.\frac{x^3}{3}\right|_0^1 + \ln|x||_1^2 = \frac{1}{3} + \ln 2 - \ln 1 = \frac{1}{3} + \ln 2.$$

4. (5 points) A particle moving along the x-axis has acceleration $a(t) = 2t + 1$ units per second per second. At time $t = 1$, it is moving left at 4 units per second. Find the average **speed** (*not average velocity*) of the particle from time $t = 0$ to time $t = 4$.

Solution: Since we have a value for the velocity function and the acceleration function, we can find the velocity function by first finding the indefinite integral of the acceleration function. We have

$$v(t) = \int a(t)\,dt = \int (2t+1)\,dt = t^2 + t + C.$$

Since $v(1) = -4$ (the particle is moving left at 4 units per second at time $t = 1$, we have

$$-4 = v(1) = 1^1 + 1 + C = 2 + C \Rightarrow C = -6,$$

and therefore $v(t) = t^2 + t - 6 = (t+3)(t-2)$. The average value of a continuous function f on the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(t)\,dt$, so to find the average speed, we need to integrate the speed function, which is $|v(t)|$. Since we are only concerned with the speed on the interval $[0, 4]$, we only need to determine $|v(t)|$ on that interval. Considering the way v factors, we have

$$|v(t)| = \begin{cases} (t+3)(2-t) & 0 \le x < 2; \\ (t+3)(t-2) & 2 \le x \le 4. \end{cases}$$

Thus, the average speed on the interval $[0,4]$ is given by

$$\begin{aligned}\frac{1}{4-0}\int_0^4 |v(t)|\,dt &= \frac{1}{4}\left(\int_0^2(-t^2-t+6)\,dt+\int_2^4(t^2+t-6)\,dt\right)\\ &= \frac{1}{4}\left(\left[-\frac{t^3}{3}-\frac{t^2}{2}+6t\right]_0^2+\left[\frac{t^3}{3}+\frac{t^2}{2}-6t\right]_2^4\right)\\ &= \frac{1}{4}\left(-\frac{8}{3}-\frac{4}{2}+12+\frac{4^3}{3}+\frac{4^2}{2}-24-\frac{2^3}{3}-\frac{2^2}{2}+12\right)\\ &= \frac{1}{4}\left(-\frac{16}{3}+4+\frac{64}{3}\right)\\ &= \frac{1}{4}\left(4+\frac{48}{3}\right)=\frac{1}{4}(4+16)=5.\end{aligned}$$

5. (5 points) Compute

$$\int_0^2\left(\sin\left(\frac{\pi x}{2}\right)+\frac{2x}{x^4+1}\right)dx.$$

Solution: We use the linearity of the integral to compute this definite integral by evaluating two separate integrals. The first is

$$\begin{aligned}\int_0^2 \sin\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi}\int_0^\pi \sin(u)\,du = -\frac{2}{\pi}\cos(u)\Big|_0^\pi &= -\frac{2}{\pi}(\cos(\pi)-\cos(0))\\ &= -\frac{2}{\pi}(-2)=\frac{4}{\pi}.\end{aligned}$$

For the second integral, we use the substitution $u=x^2$, and then $du=2xdx$. Thus, the integral is

$$\int_0^2\frac{2x}{x^4+1}\,dx=\int_0^4\frac{du}{u^2+1}=\arctan(u)|_0^4=\arctan(4)-\arctan(0)=\arctan(4).$$

Therefore, the integral we were asked to compute is $\frac{4}{\pi}+\arctan(4)$.

6. (5 points) Find the indefinite integral

$$\int\frac{x}{\sqrt[4]{x+2}}\,dx.$$

Solution: We use the substitution $u = x + 2$. Then $du = dx$ and $x = u - 2$. Therefore, the integral becomes

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \frac{u-2}{u^{1/4}}\,du = \int \left(u^{3/4} - 2u^{-1/4}\right)\,du$$

$$= \frac{u^{7/4}}{7/4} - 2\frac{u^{3/4}}{3/4} + C = \frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C.$$