MATH 1501G1/G2/G3 Test IV

Fall 2006

Name: ____________________

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Teaching Assistant and Section: ____________________

There are 6 questions on this exam on 4 pages (not counting this coverpage). Answer each question on a separate solution sheet (you may use more than one solution sheet per problem if needed). Be sure to explain your answers, as answers that are not accompanied by explanations/work may receive no credit. Place your name, section, and problem number on each solution sheet. Any solution sheet missing any of this information will **not** be graded.

You are to complete this exam completely alone, without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

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I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Failure to sign this cover page will *not* be considered evidence of academic misconduct. However, **if the cover page is not signed, three points will be deducted from your raw total score on this exam.**

Student signature: ____________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. (5 points) Let Ω be the region bounded above by the curve $y = \sin x$, below by the x-axis, on the left by $x = 0$, and on the right by $x = \pi$. Find the volume of the solid obtained by revolving Ω about the y-axis.

Solution: Because of the shape of the region, it is natural to use the shell method. The height of a cross section of Ω parallel to the y-axis is given by $\sin x$, and the radius of such a cylindrical shell is x. Therefore, the lateral surface area of a cylindrical shell is $2\pi x \sin x$. Thus, the volume of the solid we obtain is

$$\int_0^{\pi} 2\pi x \sin x \, dx.$$

We evaluate this integral using integration by parts with $u = x$ and $dv = sinx\, dx$. Then $du = dx$ and $v = -\cos x$. Thus, we have

$$\int_0^{\pi} 2\pi x \sin x \, dx = 2\pi \left(-x \cos x|_0^{\pi} - \int_0^{\pi} (-\cos x)\, dx \right) = 2\pi \left(-\pi(-1) + \sin x|_0^{\pi} \right) = 2\pi^2.$$

An alternative, shorter, approach is to realize that this region is symmetric about the line $x = \pi/2$, and thus its centroid lies on that line. That is, $\overline{x} = \pi/2$ for this region. This makes $\pi/2$ the distance from the centroid to the axis of revolution, so the Theorem of Pappus says the volume is $2\pi \cdot (\pi/2) \cdot A$ where A is the area of Ω, which a quick calculation shows is 2.

2. (5 points) Give the form of the partial fraction decomposition of

$$\frac{x^{10} - 12x^7 + 8x^4 - 9x^2 + 1}{x^3\left(x^2+1\right)^2 (x^2 - 2x + 2)(x-1)(x+3)^4}.$$

Do not solve for the constants!

Solution: The partial fraction decomposition for the given rational function has the form

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4 x + A_5}{x^2+1} + \frac{A_6 x + A_7}{(x^2+1)^2} + \frac{A_8 x + A_9}{(x^2 - 2x + 2)} + \frac{A_{10}}{x-1} + \frac{A_{11}}{x+3} + \frac{A_{12}}{(x+3)^2} + \frac{A_{13}}{(x+3)^3} + \frac{A_{14}}{(x+3)^4}.$$

3. (5 points) Evaluate $\int v^3(1-v^2)^{5/2}\, dv$.

Solution: There are, actually, several ways to evaluate this integral. What seems to be the most natural approach is a trigonometic solution, so we will give it here. You can also use a substitution or integration by parts. The trigonometric substitution we use is $v = \sin\theta$, which comes from the right triangle with hypotenuse 1 and leg lengths v and $\sqrt{1-v^2}$ with θ opposite the side of length v. With this substitution, $dv = \cos\theta\, d\theta$. Thus, the integral becomes

$$\int \sin^3\theta(1-\sin^2\theta)^{5/2}\cos\theta\, d\theta = \int \sin^3\theta(\cos^2\theta)^{5/2}\cos\theta\, d\theta = \int \sin^3\theta\cos^6\theta\, d\theta.$$

We now take $\sin^3\theta = \sin\theta(1-\cos^2\theta)$, so the integral becomes

$$\int \sin\theta(\cos^6 - \cos^8)\, d\theta = -\frac{\cos^7\theta}{7} + \frac{\cos^9\theta}{9} + C = -\frac{(1-v^2)^{7/2}}{7} + \frac{(1-v^2)^{9/2}}{9} + C.$$

4. (5 points) Let Ω be the region bounded by the curves $y = x$, $y = 0$, and $x = 1$. Find the volume of the solid obtained by revolving Ω about the line $y = -1$.

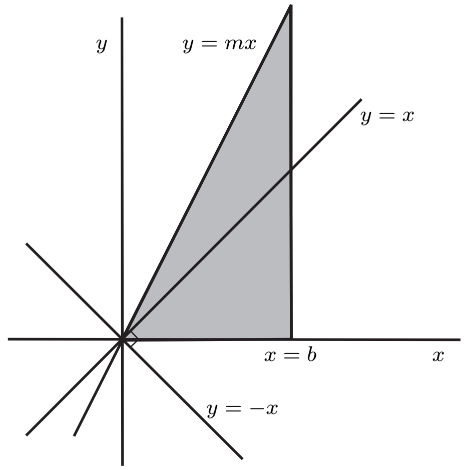
Solution: For this problem, it is most natural to use the washer method. When we revolve the given triangle about the line $y = -1$, the outer radius of the washer is $x - (-1) = x + 1$ and its inner radius is 1. Thus, the area of the washer is $\pi((x+1)^2 - 1^2)$. Therefore, the volume is

$$\int_0^1 \pi((x+1)^2 - 1)\, dx = \pi\left[\frac{(x+1)^3}{3} - x\right]_0^1 = \pi\left(\frac{8}{3} - 1 - \frac{1}{3}\right) = \frac{4\pi}{3}.$$

5. (5 points) Consider the region bounded by $y = mx$, the x-axis, and the line $x = b$, where $b > 0$ and $m > 0$ are constants.

   (a) There is a unique positive value m such that the centroid of this region is located on the line $y = x$. Find this value of m.

   (b) For the value of m above, calculate the volume of the solid obtained by revolving this region about the line $y = -x$.



Solution:

(a) Since the region is a triangle, it is easy to see that it has area $A = \frac{1}{2}b \cdot mb = mb^2/2$. Now to find the coordinates of the centroid. We find

$$\overline{x}A = \int_0^b xmx\,dx = m\frac{x^3}{3}\bigg|_0^b = \frac{mb^3}{3}.$$

This implies that $\overline{x} = 2b/3$. We also have

$$\overline{y}A = \int_0^b \frac{1}{2}(mx)^2\,dx = \frac{m^2}{2}\,\frac{x^3}{3}\bigg|_0^b = \frac{m^2b^3}{6}.$$

Therefore, we conclude that $\overline{y} = mb/3$. For the centroid to lie on $y = x$, we must have $\overline{x} = \overline{y}$, which requires $m = 2$.

(b) Since the lines $y = x$ and $y = -x$ are perpendicular, the distance from the centroid $(2b/3, 2b/3)$ to the axis of revolution is the distance from the centroid to the origin, which is $2b\sqrt{2}/3$. Thus, the Theorem of Pappus says that the volume is

$$2\pi\frac{2b\sqrt{2}}{3} \cdot \frac{2b^2}{2} = \frac{4\pi b^3\sqrt{2}}{3}.$$

6. (5 points) Evaluate $\int e^{ax}\cos(bx)\,dx$.

Solution: We use integration by parts with $u = e^{ax}$ and $dv = \cos(bx)\,dx$. Then we have $du = ae^{ax}\,dx$ and $v = \sin(bx)/b$. This gives

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}\sin(bx)}{b} - \int \frac{ae^{ax}\sin(bx)}{b}\,dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)\,dx.$$

Now we again need integration by parts. We take $u = e^{ax}$ and $dv = \sin(bx)\,dx$. Then we have $du = ae^{ax}$ and $v = -\cos(bx)/b$, so the remaining integral is

$$\int e^{ax}\sin(bx)\,dx = -\frac{e^{ax}\cos(bx)}{b} + \int \frac{ae^{ax}\cos(bx)}{b}\,dx.$$

Thus, we have

$$\begin{aligned}
\int e^{ax}\cos(bx)\,dx &= \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\left(-\frac{e^{ax}\cos(bx)}{b} + \int \frac{ae^{ax}\cos(bx)}{b}\,dx\right)\\
\Rightarrow \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos(bx)\,dx &= \frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2} + C\\
\Rightarrow \int e^{ax}\cos(bx)\,dx &= \frac{\frac{e^{ax}\sin(bx)}{b} + \frac{ae^{ax}\cos(bx)}{b^2}}{1+\frac{a^2}{b^2}} + C.
\end{aligned}$$